

**Exercise 1**

Consider two *independent* continuous RVs  $X$  and  $Y$ , whose CDFs are the following:

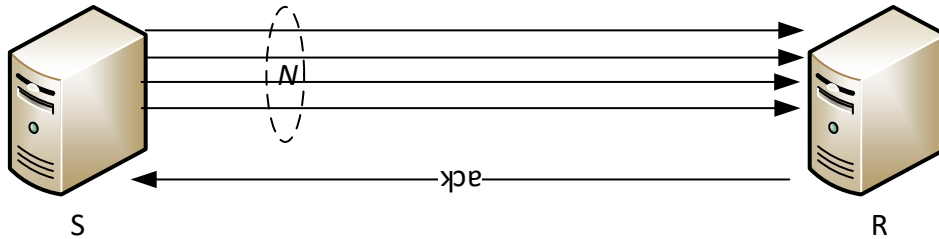
$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \leq x \leq 3 \\ 1 & x > 3 \end{cases}, \quad F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{4} & 0 \leq y \leq 2 \\ 1 & y > 2 \end{cases}$$

- 1) Compute and draw the PDF of both RVs.
- 2) Draw the region  $R$  of the Cartesian plane  $x, y$  where the JPDF  $f(x, y)$  is non null.
- 3) Identify the range of values for RV  $\frac{Y}{X}$ .
- 4) Let  $w$  be a value in the interval computed at bullet 3). Plot the region  $C_w \subseteq R$ , where inequality  $\frac{Y}{X} \leq w$  holds, on the Cartesian plane  $x, y$ . [Hint: try at least  $w = \frac{1}{2}$ ,  $w = 1$ ].
- 5) Compute  $P\{(X, Y) \in C_w\} = P\left\{\frac{Y}{X} \leq w\right\} = F_{\frac{Y}{X}}(w)$ .
- 6) Compute and draw the PDF of  $\frac{Y}{X}$ .

**Exercise 2**

Consider a network where a sender  $S$  sends packets to a receiver  $R$ , using  $N$  parallel forward links. On sending a packet,  $S$  chooses one forward link *at random* and sends the packet to it. Each forward link has a FIFO queue for buffering packets. On receipt of a packet (from any forward link),  $R$  sends an *ack* message through the return link. The transmission of both a packet and an ack take an exponential time with a mean  $\frac{1}{\mu}$ .

The above communication is subject to a *flow-control protocol*. Flow is controlled by allowing at most  $K$  packets to be un-acked at any time. Therefore, at the beginning  $S$  generates  $K$  packets, sends each to a forward link at random, and then stops and waits for an ack to come back. Whenever an ack returns,  $S$  sends a new packet. All links are error-free.



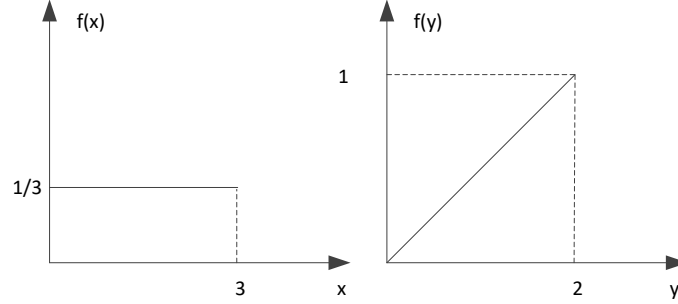
- 1) Model the above system as a Closed Jackson's Network.
- 2) Solve the routing equation system and compute the values of  $\rho_i$ .
- 3) Compute the steady-state probabilities as a function of  $N$  and  $K$  and interpret the result.

Assume  $N = 3$  and  $K = 6$

- 4) Compute  $G(M, K)$  using Buzen's algorithm
- 5) Compute the probability that  $S$  cannot send a packet due to flow control, even though all forward links are idle.
- 6) Compute the probability that all the forward links are simultaneously busy

**Exercise 1 - Solution**

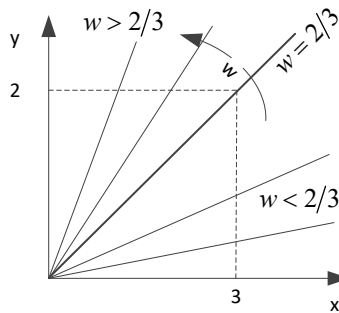
1)  $X$  is clearly a uniform RV, hence  $f(x) = \frac{1}{3}$ , for  $0 \leq x \leq 3$ , whereas  $Y$ 's PDF is  $f(y) = \frac{y}{2}$ , for  $0 \leq y \leq 2$ . The graphs of the PDFs are the following:



2) The region of the Cartesian plane where the JPDF  $f(x, y) = f(x) \cdot f(y)$  is non null is the one where the couple of RV  $(X, Y)$  may take on values, i.e. box  $R \equiv [0 \leq x \leq 3] \times [0 \leq y \leq 2]$ .

3) Since  $X$  and  $Y$  can only assume nonnegative values,  $\frac{Y}{X}$  can only assume nonnegative values. Moreover, since  $X$ 's infimum is zero, then  $\frac{Y}{X}$  takes on values in  $[0, +\infty[$ .

4) Set  $C_w$  is  $C_w = \{(x, y): (x, y) \in R, \frac{y}{x} \leq w\}$ , and it includes the area *below* each of the lines drawn in the figure, whose slope is  $w$ . Set  $C_w$  covers a triangle if  $0 \leq w \leq \frac{2}{3}$ , and a trapezoid if  $w > \frac{2}{3}$ .



5) It is  $P\{(X, Y) \in C_w\} = \int_{C_w} \int f(x, y) dx dy$ , and  $f(x, y) = f(x) \cdot f(y) = \frac{y}{6}$  for  $(x, y) \in R$ . We thus need to integrate  $f(x, y)$  on set  $C_w$ . We need to distinguish the cases  $0 \leq w \leq \frac{2}{3}$  and  $w > \frac{2}{3}$ .

a)  $0 \leq w \leq \frac{2}{3}$ . We integrate  $y$  from 0 to  $w \cdot x$ , and  $x$  from 0 to 3.

$$P\{(X, Y) \in C_w\} = \int_{C_w} \int \frac{y}{6} dx dy = \int_0^3 \left[ \int_0^{w \cdot x} \frac{y}{6} dy \right] dx = \int_0^3 \left[ \frac{w^2 x^2}{12} \right] dx = \frac{3}{4} w^2$$

b)  $w > \frac{2}{3}$ . In this case, it is quicker to use the complement, i.e. to exclude the upper triangular region.

We need to invert the order of integration of the variables.

$$\begin{aligned} P\{(X, Y) \in C_w\} &= \int_{C_w} \int \frac{y}{6} dx dy = 1 - \int_0^2 \int_0^{\frac{y}{w}} \frac{y}{6} dx dy = 1 - \int_0^2 \frac{y^2}{6w} dy = 1 - \left[ \frac{y^3}{18w} \right]_0^2 \\ &= 1 - \frac{4}{9w} \end{aligned}$$

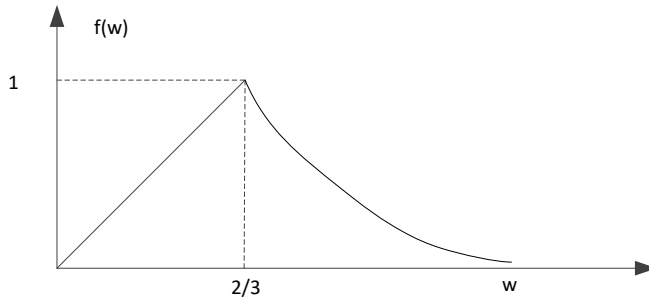
6) Since  $P\{(X, Y) \in C_w\} = P\left\{\frac{Y}{X} \leq w\right\} = F_{\frac{Y}{X}}(w)$ , we have:

$$F_{\frac{Y}{X}}(w) = \begin{cases} \frac{3}{4}w^2 & 0 \leq w \leq \frac{2}{3} \\ 1 - \frac{4}{9w} & w > \frac{2}{3} \end{cases}.$$

Note that  $F_{\frac{Y}{X}}(w)$  is continuous, and  $\lim_{w \rightarrow +\infty} F_{\frac{Y}{X}}(w) = 1$ . From that, we get:

$$f_{\frac{Y}{X}}(w) = \begin{cases} \frac{3}{2}w & 0 \leq w \leq \frac{2}{3} \\ \frac{4}{9w^2} & w > \frac{2}{3} \end{cases}$$

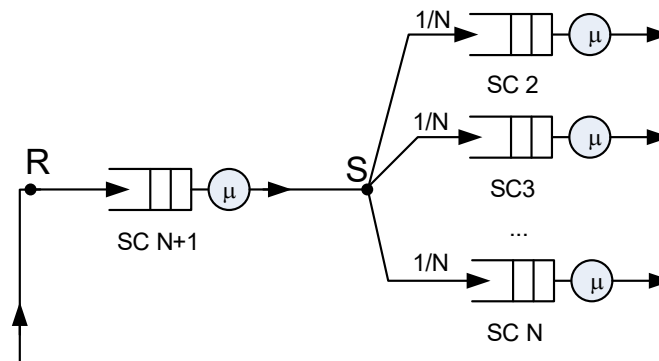
The graph of the PDF is the following:



## Exercise 2 - Solution

1) Let SCs 1 to  $N$  be each output interface at  $S$ , and let  $SC N + 1 = M$  be the output interface at  $R$ .

Then a CJN model of the above system is the following:



2) The routing matrix is the following:  $\underline{\Pi} = \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ \frac{1}{N} & \frac{1}{N} & \dots & 0 \end{bmatrix}$

And the equations are  $\lambda_i = \frac{\lambda_{N+1}}{N}$ ,  $1 \leq i \leq N$ . It is fairly obvious that solutions are of the form  $\underline{e}^T = [e, e, \dots, e, N \cdot e]^T$ . Hence, we select  $e = \mu$  and obtain  $\underline{\rho}^T = [1, 1, \dots, 1, N]^T$ .

3) The SS probabilities are  $P(n_1, n_2, \dots, n_{N+1})|_{\sum_{i=1}^{N+1} n_i = K} = \frac{1}{G(N+1, K)} \cdot \prod_{i=1}^{N+1} \rho_i^{n_i} = \frac{1}{G(N+1, K)} \cdot N^{n_{N+1}}$ .

Since the above expression only depends on  $n_{N+1}$ , this means that any distribution of packets on the forward link queues having the same number of outstanding acks  $n_{N+1}$  is equally likely.

4) Buzen's algorithm yields the following table:

| rho  | 1 | 1 | 1  | 3    |
|------|---|---|----|------|
| s.c. | 1 | 2 | 3  | 4    |
| jobs |   |   |    |      |
| 0    | 1 | 1 | 1  | 1    |
| 1    | 1 | 2 | 3  | 6    |
| 2    | 1 | 3 | 6  | 24   |
| 3    | 1 | 4 | 10 | 82   |
| 4    | 1 | 5 | 15 | 261  |
| 5    | 1 | 6 | 21 | 804  |
| 6    | 1 | 7 | 28 | 2440 |

Hence  $G(4,6) = 2440$ .

Note that, due to the findings at point 3),  $G$  could also be computed using the normalization condition instead of Buzen's algorithm:

$$\sum_{\bar{n} \in e} P(n_1, n_2, \dots, n_{N+1}) = \sum_{\bar{n} \in e} \left( \frac{1}{G(N+1, K)} \cdot N^{n_{N+1}} \right) = 1, \text{ from which:}$$

$$G(N+1, K) = \sum_{\bar{n} \in e} N^{n_{N+1}} = \sum_{n_{N+1}=0}^K \binom{K - n_{N+1} + N - 1}{N - 1} N^{n_{N+1}}.$$

The binomial coefficient is the number of ways to split  $K - n_{N+1}$  jobs (i.e., those that are not on SC  $N+1$ ) among  $N$  SCs, and can be found on the course notes. The two methods obviously yield the same result.

5) The probability that the sender cannot send a packet due to flow control is the probability that  $K$  acks are queued at the return link, hence  $p_{block} = \frac{1}{G(N+1, K)} \cdot N^K = \frac{3^6}{2440} = \frac{729}{2440} \approx 0.3$

6) The probability that all the forward links are simultaneously busy is:

$$p_{busy} = \frac{G(M, K-N)}{G(M, K)} \prod_{i=1}^N \rho_i^1 = \frac{G(4,3)}{G(4,6)} = \frac{82}{2440} \approx 0.033.$$

