

Exercise 1

Consider the following function:

$$f(x) = \begin{cases} \frac{k}{c} \cdot \left(\frac{x}{c}\right)^{k-1} & 0 \leq x \leq c \\ 0 & \text{otherwise} \end{cases}.$$

- 1) Express the conditions under which $f(x)$ is a PDF
- 2) Compute the CDF $F(x)$, draw it for $k = 1$ and $k = 2$, and discuss what happens when $k \rightarrow \infty$.
- 3) Compute the mean value and the median. Justify the result when $k = 1$ and $k \rightarrow \infty$.
- 4) Compute the variance. Justify the result.
- 5) Consider RV $S_j = \sum_{i=1}^j X_i$, where the X_i s are IID distributed according to $f(x)$. Compute the coefficient of variation C of S_j . Justify the result.

Exercise 2

An airport check-in has exponential arrivals, with a rate λ , and two desks. To optimize staff, the airport policy is that one of the desks is kept unmanned until there are *more than* T passengers. Assume that the rate of service of each desk is identical and equal to μ , and that passengers are served individually.

- 1) Draw the CTMC
- 2) Identify the stability condition¹ and state explicitly its dependence on T .
- 3) compute the steady-state probabilities as a function of T .
- 4) Compute the probability that an arriving job finds exactly T jobs in the system, and compute under what conditions T is the most likely number of jobs that an arriving job will see. Justify the result.
- 5) Compute the average number of jobs in the system when $\lambda = \mu$, as a function of T . Justify the result.

¹ Computations are required. Educated guesses will not be accepted.

Exercise 1 - Solution

1) The conditions that must hold are that:

a) the function must be positive, hence $k > 0$

b) its integral must be equal to 1, i.e.: $\int_{-\infty}^{+\infty} f(x)dx = \frac{k}{c^k} \cdot \int_0^c x^{k-1}dx = \frac{k}{c^k} \cdot \left[\frac{x^k}{k} \right]_0^c = 1$.

Therefore, the only required condition is that $k > 0$.

2) The CDF can be easily obtained as: $F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{c}\right)^k & 0 \leq x \leq c \\ 1 & x > c \end{cases}$

As k grows large, the CDF tends to become more and more similar to a step function in $x = c$. Note that this can be interpreted as the distribution of the *maximum* of k IID RVs $\sim U(0,c)$, as seen during classes.

3) The mean value is $E[X] = \int_0^c x \cdot f(x)dx = \int_0^c k \cdot \left(\frac{x}{c}\right)^k dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+1}}{k+1} \right]_0^c = \frac{k \cdot c}{k+1}$.

The median is instead obtained by equating $F(x) = 0.5$, i.e., $\left(\frac{x}{c}\right)^k = \frac{1}{2}$, i.e. $x_{0.5} = \frac{c}{\sqrt[k]{2}}$. When $k = 1$ the distribution is uniform, hence symmetric, thus $E[X] = x_{0.5}$. As k grows large, it is again $E[X] = x_{0.5} = c$, since the distribution becomes a step function.

4) The mean square value is: $E[X^2] = \int_0^c x^2 \cdot f(x)dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+2}}{k+2} \right]_0^c = \frac{k \cdot c^2}{k+2}$.

Hence the variance is: $\sigma^2 = E[X^2] - E[X]^2 = \frac{k \cdot c^2}{k+2} - \frac{k^2 \cdot c^2}{(k+1)^2} = \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}$.

The variance depends on the square of c , since the latter is the range of the variable. Moreover, it decreases with k . This makes perfect sense, since, as k grows large, the variability of the distribution decreases as well, as already observed. When $k = 1$, we obtain again the well-known variance of a uniform variable.

5) The mean and variance are $j \cdot \frac{k \cdot c}{k+1}$ and $j \cdot \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}$ respectively. Thus, it is:

$$C = \frac{\sqrt{j \cdot \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}}}{j \cdot \frac{k \cdot c}{k+1}} = \frac{\sqrt{j \cdot \frac{k}{(k+2)}}}{j \cdot k} = \frac{1}{\sqrt{j \cdot k \cdot (k+2)}}$$

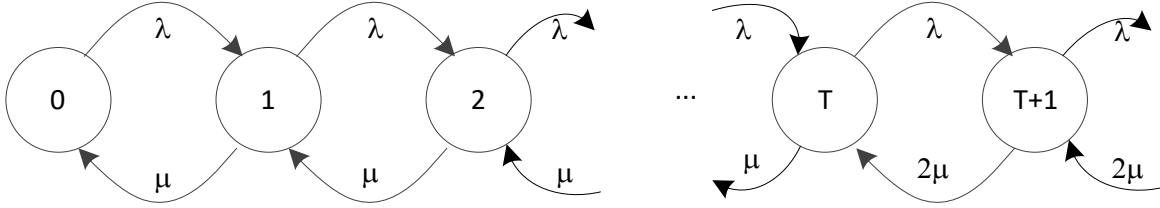
The result is in accord with the Central Limit Theorem, which is applicable since variables are IID with finite mean and variance. The CoV decreases with the number of variables, and it decreases

faster if k is large. In fact, as k grows large, the mean converges, but the variance goes to zero. When k is large, the following approximation holds:

$$C = \frac{1}{\sqrt{j \cdot k \cdot (k+2)}} \approx \frac{1}{k} \cdot \frac{1}{\sqrt{j}}.$$

Exercise 2 - Solution

1) The CTMC is as follows:



2) The stability condition can be easily inferred to be *independent of T* . In fact, the fact that the service rate is μ up to state T is irrelevant, as long as the remaining states (up to infinity) have a different service rate 2μ . This said, the stability condition can only be $\lambda < 2\mu$. The same result should emerge from the computations as well. In fact, the local balance equations yield:

$$p_i = \begin{cases} \frac{\lambda}{\mu} p_{i-1} & i \leq T \\ \frac{\lambda}{2\mu} p_{i-1} & i > T \end{cases}, \text{ hence } p_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \cdot p_0 & i \leq T \\ \left(\frac{\lambda}{2\mu}\right)^{i-T} \cdot \left(\frac{\lambda}{\mu}\right)^T \cdot p_0 & i > T \end{cases}$$

Therefore, the normalization condition is as follows:

$$p_0 \cdot \left[\sum_{i=0}^T \left(\frac{\lambda}{\mu}\right)^i + \sum_{i=T+1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^T \cdot \left(\frac{\lambda}{2\mu}\right)^{i-T} \right] = 1$$

Which can be rewritten as:

$$p_0 \cdot \left[\sum_{i=0}^T \left(\frac{\lambda}{\mu}\right)^i + \left(\frac{\lambda}{\mu}\right)^T \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{2\mu}\right)^j \right] = 1.$$

The first sum is always finite, and the second one converges if and only if $\lambda < 2\mu$. Call $u = \frac{\lambda}{2\mu}$.

3) We need to treat separately the two cases $\lambda = \mu$, $\lambda \neq \mu$ (because of the first sum).

Case 1: $\lambda = \mu$

In this case, we have $p_0 \cdot \left[(T+1) + \frac{u}{1-u} \right] = 1$, hence:

$$p_0 = \frac{1-u}{T \cdot (1-u) + 1}, \quad p_i = \begin{cases} p_0 & i \leq T \\ u^{i-T} \cdot p_0 & i > T \end{cases}$$

In this case, all s.s. probabilities are equal up to state T . This should not surprise us, since the states at the leftmost part of the diagram have equal transition rates to the left and to the right.

Case 2: $\lambda \neq \mu$

We get: $p_0 \cdot \left[\frac{1-(2u)^{T+1}}{1-2u} + (2u)^T \cdot \frac{u}{1-u} \right] = 1$, i.e. $p_0 = \frac{1}{\frac{1-(2u)^{T+1}}{1-2u} + \frac{(2u)^T \cdot u}{1-u}}$

The above expression can be simplified to $p_0 = \frac{(1-2u) \cdot (1-u)}{(1-u) - u \cdot (2u)^T}$.

$$p_i = \begin{cases} (2u)^i \cdot p_0 & i \leq T \\ u^{i-T} \cdot (2u)^T \cdot p_0 & i > T \end{cases}$$

4) This system has constant arrival rates, hence it possesses the PASTA property, i.e., $r_T = p_T$. The answer is, thus, to find the condition under which $p_K > p_i$, $i \neq T$. Note that, if $\lambda = \mu$, the above condition is false (all states up to T have the same probability), therefore we can safely assume $\lambda \neq \mu$.

In this case, we observe that $u^{i-T} \cdot (2u)^T \cdot p_0$ is by definition a decreasing sequence (otherwise the system would not be stable). On the other hand, $(2u)^i \cdot p_0$ is an increasing sequence if $\mu < \lambda < 2\mu$. Therefore, state T will be the most likely number of jobs if and only if $\mu < \lambda < 2\mu$.

5) When $\lambda = \mu$ we have:

$$p_0 = \frac{1-u}{T \cdot (1-u) + 1}, p_i = \begin{cases} p_0 & i \leq T \\ u^{i-T} \cdot p_0 & i > T \end{cases}$$

Hence,

$$\begin{aligned} E[N] &= \sum_{i=0}^T i \cdot p_0 + \sum_{i=T+1}^{+\infty} i \cdot p_0 \cdot u^{i-T} \\ &= p_0 \cdot \left[\sum_{i=0}^T i + \sum_{j=1}^{+\infty} (T+j) \cdot u^j \right] \\ &= p_0 \cdot \left[\frac{T \cdot (T+1)}{2} + T \frac{u}{1+u} + \frac{u}{(1-u)^2} \right] \end{aligned}$$

By substituting $u = \frac{1}{2}$ we get

$$E[N] = \frac{1}{T+2} \cdot \left[\frac{T \cdot (T+1)}{2} + (T+2) \right] = \frac{T}{2} \cdot \frac{T+1}{T+2} + 1$$

The result makes sense. In fact, if $u = \frac{1}{2}$, all the states up to T are equally likely (hence the first term), but states *above* T have non null probability. Therefore, the result must be larger than $T/2$, which would be the result if those states did not exist.