PECSN, 18/6/2026

Exercise 1

Consider the following PDF:

$$f(x) = \begin{cases} -\alpha x^2 + 1 & x \in \left[-\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\alpha}}\right] \\ 0 & \text{otherwise} \end{cases}$$

1) Compute α such that $f(x)$ is a PDF;
2) Compute $E[X]$ and $Var(X)$;
3) Consider now $Y = |X|$. Compute its PDF, mean value and mean square value.

Exercise 2

A network switch has N available buffer slots. Its packets arrive at a rate $\lambda_k = \lambda \cdot \frac{N-k}{N(k+1)}$, where k is the number of packets in the switch. Let μ be the switch's service rate.

1) Model the system as a CTMC, compute its stability condition and its steady-state probabilities;
2) Compute the average arrival rate of the packets;
3) Compute the SS probabilities observed by an arriving packet;
4) Compute the mean number of packets in the system.
5) Assume that $\lambda = N \cdot \mu$. What is the system state that a random observer will observe with the highest probability? (assume N is even). Why?

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Exercise 1 – solution

In order to compute parameter α, one needs to impose normalization, i.e.

$$\int_{-\frac{1}{\sqrt{\alpha}}}^{\frac{1}{\sqrt{\alpha}}} (-\alpha x^2 + 1) dx = 1$$

From the above, we get:

$$\left[-\alpha\frac{x^3}{3} + x\right]_{-\frac{1}{\sqrt{\alpha}}}^{\frac{1}{\sqrt{\alpha}}} = -\frac{\alpha}{3\alpha\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}} - \left(\frac{\alpha}{3\alpha\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha}}\right) = -\frac{2}{3\sqrt{\alpha}} + \frac{2}{\sqrt{\alpha}} = \frac{4}{3\sqrt{\alpha}} = 1$$

Hence $\alpha = 16/9$.

A straightforward plot shows $f(x)$ to be an even function, hence $E[X] = 0$. This also implies that:

$$Var(X) = E[X^2] = \int_{-3/4}^{3/4} x^2 \left(-\frac{16}{9}x^2 + 1\right) dx = \cdots = \frac{9}{80}$$

And that $f_Y(x) = 2f(x)$, i.e.:

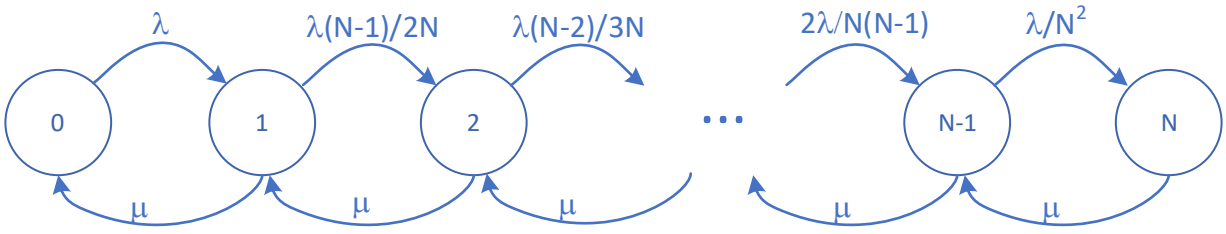
$$f_Y(x) = \begin{cases} -\frac{32}{9}x^2 + 2 & x \in \left[0, \frac{3}{4}\right] \\ 0 & \text{otherwise} \end{cases}$$

Now, we have:

$$E[Y] = 2\int_0^{3/4} x\left(-\frac{16}{9}x^2 + 1\right) dx = 2\left[-\frac{4x^4}{9} + \frac{x^2}{2}\right]_0^{3/4} = 2\left[-\frac{9}{64} + \frac{9}{32}\right] = \frac{9}{32}$$

$$E[Y^2] = 2\int_0^{3/4} x^2\left(-\frac{16}{9}x^2 + 1\right) dx = 2\left[-\frac{16x^5}{9\cdot 5} + \frac{x^3}{3}\right]_0^{3/4} = 2\left[-\frac{16\cdot 3^5}{9\cdot 5\cdot 4^5} + \frac{3^3}{4^3\cdot 3}\right] = \left[\frac{3^2}{5\cdot 4^2}\right] = \frac{9}{80}$$

Exercise 2 – solution



This system is always stable, since it has a finite number of states.

Using global equations, one finds immediately that

$$p_k = \binom{N}{k}\left(\frac{\lambda}{N\mu}\right)^k p_0, \quad 0 \le k \le N$$

Call $\alpha = \frac{\lambda}{N\mu}$ for simplicity. Normalization reads:

$$p_0 \sum_{k=0}^{N} \binom{N}{k} \alpha^k = p_0 \sum_{k=0}^{N} \binom{N}{k} \alpha^k 1^{N-k} = p_0 (1+\alpha)^N = 1$$

Hence,

$$p_0 = \frac{1}{(1+\alpha)^N}$$

And

$$p_k = \binom{N}{k} \frac{\alpha^k}{(1+\alpha)^N}, \quad 0 \le k \le N$$

This system is non-PASTA. Therefore, $r_k \neq p_k$. It is $r_k = p_k \cdot \frac{\lambda_k}{\bar{\lambda}}$.

Now,

$$\bar{\lambda} = \gamma = \mu(1-p_0) = \mu \left[1 - \frac{1}{(1+\alpha)^N}\right]$$

Hence:

$$r_k = \binom{N}{k} \frac{\alpha^k}{(1+\alpha)^N} \cdot \frac{\lambda \cdot \frac{N-k}{N(k+1)}}{\mu \left[1 - \frac{1}{(1+\alpha)^N}\right]} = \binom{N}{k} \frac{N-k}{k+1} \cdot \frac{\alpha^{k+1}}{(1+\alpha)^N} \cdot \frac{1}{1 - \frac{1}{(1+\alpha)^N}}$$

$$= \binom{N}{k+1} \frac{\alpha^{k+1}}{(1+\alpha)^N - 1}, \quad 0 \le k < N$$

The mean number of packets is easier to compute if one observes that

$$p_k = \binom{N}{k} \frac{\alpha^k}{(1+\alpha)^N} = \binom{N}{k} \left(\frac{\alpha}{1+\alpha}\right)^k \left(\frac{1}{1+\alpha}\right)^{N-k}$$

Which is a binomial PMF with N trials and a success probability $p = \frac{\alpha}{1+\alpha}$. Therefore, its mean value is:

$$E[N] = N \frac{\alpha}{1+\alpha}$$

The mean response time can be computed through Little's law, and it is equal to:

$$E[R] = \frac{E[N]}{\gamma} = N \frac{\alpha}{1+\alpha} \cdot \frac{1}{\mu \left[1 - \frac{1}{(1+\alpha)^N}\right]}$$

When $\alpha = 1$, it is $E[N] = N/2$. Since p_k is a binomial distribution, and N is even, $N/2$ is integer and therefore $k^* = N/2$ is an observable state. A binomial PMF peaks "around" its mean value $E[N]$. Therefore $k^* = N/2$ is the state that will be observed by a random observer with the highest probability.