Exercise 1

Let X, Y be two RVs whose JPDF is:

$$f(x,y) = \begin{cases} c(x+y) & 0 \le x \le 1, 0 < y \le 1-x \\ 0 & \text{otherwise} \end{cases}$$

1) Find constant c;
2) Compute the PDFs of the single RVs, and draw them qualitatively;
3) Are the RVs independent? Justify your answer;
4) Compute the CDF of $Z = X + Y$

Exercise 2

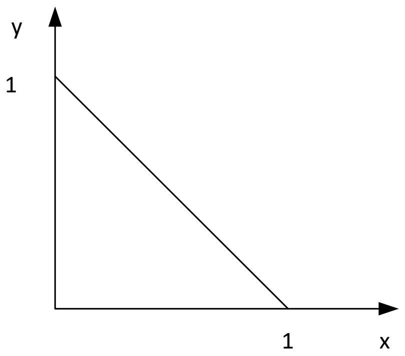
A computing system can handle several tasks. A processing element called *dispatcher* processes tasks initially, at a rate μ, and then sends them to either of two processing chains, with a probability π and $1 - \pi$, respectively. The first processing chain consists of two processing elements in tandem, each one having a rate μ. The second chain has a single processing element with a rate μ. After passing through the processing chains, tasks are completed.

Assume that the system has an infinite supply of tasks available, so that, whenever a task is completed, another one pops up immediately. Call K the number of tasks being handled by the system during the analysis.

1) Model the system as a queueing network. Compute its routing matrix, find the vector of arrival rates and the vector of ρs.
2) Compute the generic formula for the SS probabilities.
3) Find the normalizing constant using Buzen's algorithm when $\pi = 0.5$ and $K = 4$.
4) Compute the utilization of each processing element and the task throughput.
5) Compute the mean response time at each processing element and the mean circuit time.

Exercise 1 – Solution

1) The JPDF is defined in a triangle as follows:



Constant c can be computed via normalization:

$$\int_0^1 \left[\int_0^{1-x} c(x+y)\ dy\right] dx = \cdots = \frac{c}{3}$$

Therefore, $c = 3$.

2) The PDFs of the two RVs are:

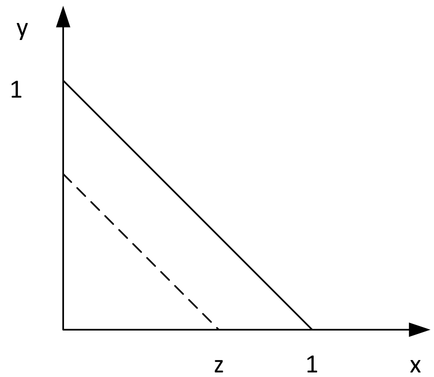
$$f_X(x) = \int_0^{1-x} 3(x+y)\ dy = \cdots \frac{3}{2}(1-x^2), \quad f_Y(y) = \int_0^{1-y} 3(x+y)\ dx = \cdots \frac{3}{2}(1-y^2)$$

Both are defined in $[0,1]$, and are arcs of parabola with the concavity facing downwards.

3) One can check that $f(x,y) \neq f_X(x) \cdot f_Y(y)$, hence the two RVs are not independent.

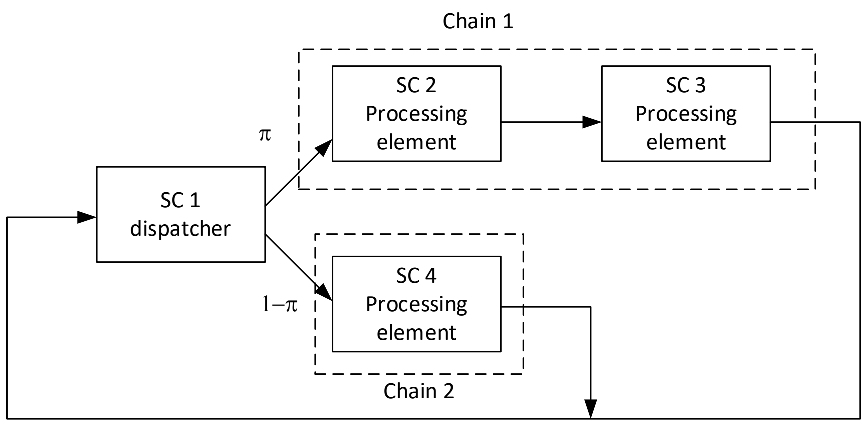
4) RV Z is defined in $[0,1]$ as well. $F_Z(z)$ can be found by integrating the JPDF in the triangle bounded by the dashed line. Therefore, it is:

$$F_Z(z) = \int_0^z \left[\int_0^{z-x} 3(x+y)\ dy\right] dx = \cdots = z^3$$



Exercise 2 – Solution

1) This system can be modeled as a CJN with 4 service centers, as in the following drawing:



Each SC is an M/M/1 node with a serving rate μ. With the above numbering, the routing matrix and the vectors are:

$$\underline{\mathbf{\Pi}} = \begin{bmatrix} 0 & \pi & 0 & 1-\pi \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{e}} = [e, \pi e, \pi e, (1-\pi)e]^T$$

Choosing $e = \mu$, we obtain:

$$\underline{\boldsymbol{\rho}} = [1, \pi, \pi, 1-\pi]^T$$

And the formula for the SS probabilities is:

$$p_{\underline{n}} = p(n_1, n_2, n_3, n_4) = \frac{1}{G(M,K)} \cdot \pi^{n_2+n_3} \cdot (1-\pi)^{n_4}$$

Where $n_1 + n_2 + n_3 + n_4 = K$.

We run Buzen's algorithm with the above numbers and we obtain the following:

SC	1	2	3	4
ρ	1	1/2	1/2	1/2
jobs				
0	1	1	1	1
1	1	3/2	2	5/2
2	1	7/4	11/4	16/4
3	1	15/8	26/8	42/8
4	1	31/16	57/16	99/16

Hence $G(M,K) = 99/16$.

$$p_{\underline{n}} = p(n_1, n_2, n_3, n_4) = \frac{16}{99} \cdot \frac{1}{2^{n_2+n_3+n_4}}.$$

The utilization at each SC i is $U_i = \rho_i \cdot \frac{G(M,K-1)}{G(M,K)} = \frac{42}{8} \cdot \frac{16}{99} \rho_i = \frac{84}{99} \rho_i$, i.e.:

$$U_1 = \frac{84}{99}, \quad U_2 = U_3 = U_4 = \frac{42}{99}$$

The task throughput is:

$$\gamma = \gamma_1 = U_1 \cdot \mu = \frac{84}{99} \cdot \mu.$$

The throughput of the other SCs is:

$$\gamma_2 = \gamma_3 = \gamma_4 = U_2 \cdot \mu = \frac{42}{99} \cdot \mu.$$

The mean number of jobs at each SC is

$$E[N_i] = \frac{1}{G(M,K)} \cdot \left[\sum_{h=1}^{K} {\rho_i}^h \cdot G(M, K-h) \right],$$

hence:

$$E[N_1] = \frac{16}{99} \cdot \left[\sum_{h=1}^{K} G(M, K-h) \right] = \frac{16}{99} \cdot \frac{51}{4} = \frac{204}{99} = \frac{68}{33}$$

$$E[N_2] = E[N_3] = E[N_4] = \frac{16}{99} \cdot \left[\sum_{h=1}^{K} \left(\frac{1}{2}\right)^h \cdot G(M, K-h) \right] = \frac{16}{99} \cdot 4 = \frac{64}{99}$$

The response times at the SCs are:

$$E[R_1] = \frac{E[N_1]}{\gamma_1} = \frac{68}{33} \cdot \frac{99}{84} \cdot \frac{1}{\mu} = \frac{17}{7} \cdot \frac{1}{\mu}$$

$$E[R_2] = E[R_3] = E[R_4] = \frac{64}{99} \cdot \frac{99}{42} \cdot \frac{1}{\mu} = \frac{32}{21} \cdot \frac{1}{\mu}$$

The mean circuit time is:

$$E[C] = E[R_1] + \pi \cdot (E[R_2] + E[R_3]) + (1-\pi) \cdot E[R_4]$$

$$E[C] = \frac{51}{21} \cdot \frac{1}{\mu} + \frac{1}{2} \cdot \left(\frac{64}{21} \cdot \frac{1}{\mu}\right) + \frac{1}{2} \cdot \frac{32}{21} \cdot \frac{1}{\mu} = \frac{99}{21} \cdot \frac{1}{\mu} = \frac{33}{7} \cdot \frac{1}{\mu}$$