Exercise 1

1) Flip a fair coin twice. What is the probability that you get two heads (HH)? What is the probability that you get heads followed by tails (HT)? Are these probabilities the same?

2) Flip a fair coin repeatedly until you get heads and tails in a row (HT). What is the probability that it takes n flips to win?

3) Flip a fair coin repeatedly until you get two heads in a row (HH). What is the probability that it takes n flips to win? (*suggestion*: go all the way up to $n = 8$ before making conclusions).

4) Based on the answers to points 2 and 3, is the probability of a *large* value of n equal in the two cases? If it is not, which probability is the highest?

5) Player A and B play the following game: they flip a coin repeatedly until either HH occurs (A wins) or HT occurs (B wins). Is the game fair (i.e., are the two players equally likely to win)?

Exercise 2

A service is hosted on a system having K identical servers. A job dispatcher routes each arriving job to an idle server, if there exists one, and *rejects it* otherwise. Assume that the service time and interarrival time are exponentially distributed RVs, whose rates are μ, λ respectively.

1) Draw the CTMC (or transition rate diagram).
2) Compute the stability condition and an expression for the steady-state probabilities.
3) Find under what conditions the PMF of the steady-state probability is:
    a. strictly increasing, i.e., $p_{j+1} > p_j$, $0 \leq j < K$.
    b. strictly decreasing.
    c. Neither of the above.

    Explain your findings.

4) Compute the loss probability
5) Compute the mean number of busy servers and, from that, the mean response time. Discuss both expressions when $K \to \infty$.
6) Compute the steady-state probabilities and the performance indexes for $K = 1$. Discuss the result.

Exercise 1 – Solution

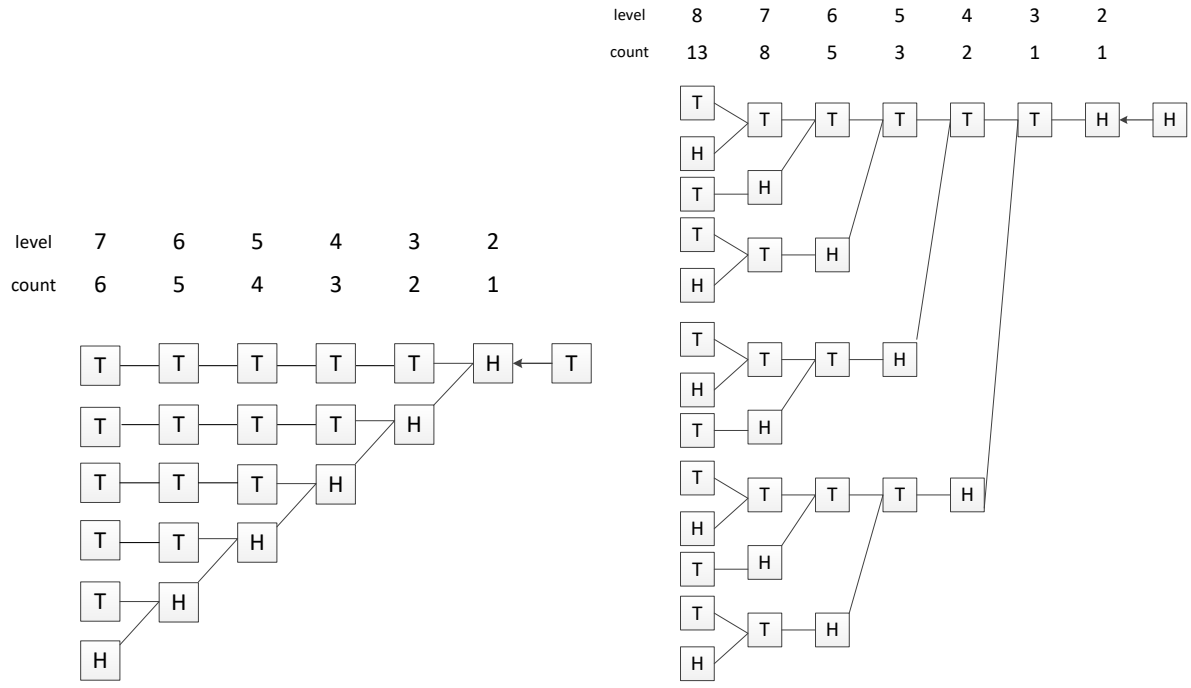
1. The two probabilities are obviously the same, and each one is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

2. This is clearly a uniform probability model. The number of n-sequences is 2^n, whereas the number of *good* n-sequences can be computed by constructing a tree. The topmost two levels of the tree are HT, and every time a level *j* is added, only feasible sequences are generated (i.e., you only add a child T to a parent T, whereas you add both children H and T to a parent H). The resulting tree is shown in the left figure below. The number of *good* n-sequences increases by one at each step, i.e., it grows linearly and is equal to $n - 1$. Hence:

$P\{N_{HT} = n\} = \frac{n-1}{2^n}, n \geq 2$

3. By the same argument we find that the number of *good* n-sequences is different. The highest two levels of the tree are HH, and every time a level *j* is added, you only add a child T to a parent H, whereas you add both children H and T to a parent T. The resulting tree is shown in the right figure below. The number of *good* n-sequences is the $n - 1$ number in the Fibonacci sequence S_{n-1}. Therefore:

$P\{N_{HH} = n\} = \frac{S_{n-1}}{2^n}, n \geq 2$

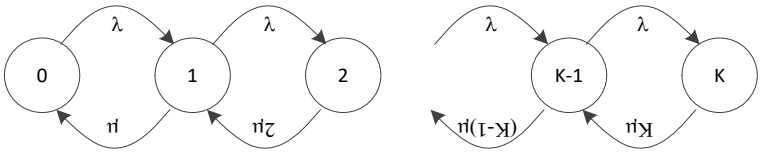


4. Given the above, we can observe that $S_{n-1} > n - 1$ starting from $n = 6$ (The fact that the Fibonacci sequence is superlinear is well known). Therefore, the probability of observing long sequences is *larger* if you bet on HH.

5. Counterintuitively, the game *is* fair, despite the fact that you observe longer sequences more often with HH than with HT. In fact, the *only* n-sequence that can go on indefinitely without either player winning is {TT...TT}. In fact, as soon as *one* H appears, the winner is decided by the next flip. So, there is only *one* sequence that leads to the decisive flip, and it is {TTT…TT}H. After that, both players have the same 50% chance to win.

Exercise 2 – Solution

1) The diagram is shown below



2) The system is finite, hence always stable. The steady-state probabilities can be computed by writing the local balance equations: $\lambda \cdot p_j = (j+1) \cdot \mu \cdot p_{(j+1)}$, $0 \leq j \leq K-1$. From the above, we quickly obtain:

$$p_j = \frac{1}{j!} \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot p_0, \;\; 0 \leq j \leq K$$

Call $u = \frac{\lambda}{\mu}$. By imposing the normalization condition, we get:

$$p_0 = \frac{1}{\sum_{i=0}^{K} \frac{u^i}{i!}} \quad p_j = \frac{\frac{u^j}{j!}}{\sum_{i=0}^{K} \frac{u^i}{i!}}$$

3) $p_{j+1} > p_j \Leftrightarrow \frac{\frac{u^{j+1}}{(j+1)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} > \frac{\frac{u^j}{(j)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} \Leftrightarrow u > j+1$. Since the above must hold for every j up to $K-1$ included, the required condition is $u > K$. This can be explained by observing that, under this condition, the arrival rate is such that the serving capacity of the system (which is $K\mu$ under a heavy load) is smaller than the arrival rate. Using the same reasoning, we find that the PMF is decreasing when $u < 1$. This means that a single server has enough capacity to serve the arrivals, hence it is increasingly more unlikely that more servers will be busy. When $1 \leq u \leq K$, the PMF is neither (strictly) increasing nor (strictly) decreasing.

4) The loss probability is

$$p_K = \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K} \frac{u^i}{i!}}$$

5) The mean number of busy servers is:

$$E[N] = \sum_{j=1}^{K} j \cdot p_j = \frac{\sum_{j=1}^{K} j \frac{u^j}{j!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = \frac{u \cdot \sum_{j=1}^{K} \frac{u^{j-1}}{(j-1)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = u \cdot \frac{\left[\sum_{j=0}^{K} \frac{u^j}{j!} - \frac{u^K}{K!}\right]}{\sum_{i=0}^{K} \frac{u^i}{i!}} = u \cdot (1 - p_K)$$

The mean response time can be obtained through Little's theorem, using the *mean* arrival rate $\overline{\lambda} = \lambda \cdot (1 - p_K)$. Thus, the average response time is $E[R] = \frac{E[N]}{\overline{\lambda}} = \frac{u \cdot (1-p_K)}{\lambda \cdot (1-p_K)} = \frac{1}{\mu}$. In fact, there is never any queueing in this system.

When $K \to \infty$ the system becomes an $M/M/\infty$. For this system, the mean number of busy servers is in fact $u = \frac{\lambda}{\mu}$. This can only happen if $\lim_{K\to\infty} p_k = 0$. Note that this condition is necessary of *any* PMF with an infinite support, hence must hold for this PMF as well. We can check it explicitly:

$$\lim_{K\to\infty} p_K = \lim_{K\to\infty} \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = e^{-u} \cdot \lim_{K\to\infty} \frac{u^K}{K!} = 0$$

6) When $K = 1$ we have

$$p_0 = \frac{1}{\sum_{i=0}^{K} \frac{u^i}{i!}} = \frac{1}{1+u}, \quad p_1 = \frac{\frac{u^1}{1!}}{1+u} = \frac{u}{1+u}$$

It is $p_K = p_1 = \frac{u}{1+u}$, $E[N] = p_1 = \frac{u}{1+u}$, and also

$$E[N] = u \cdot (1 - p_K) = u \cdot \left(1 - \frac{u}{1+u}\right) = u \cdot \left(\frac{1}{1+u}\right) = \frac{u}{1+u}$$

$$E[R] = \frac{1}{\mu}.$$

In this case, the system is an $M/M/1/1$ (or a 1-buffer), and the reader can easily check that the above expressions match those of such a system.