PECSN, 23/07/2025

Exercise 1

Consider the following function:

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Where α, β are *positive* constants.

1) Determine under what conditions $F(x)$ is a CDF.

Assume from now on that we are in the above conditions.

2) Compute the PDF of RV X, whose CDF is $F(x)$.
3) Determine under what further conditions $E[X]$ is finite.
4) Assuming $\beta = 5$, compute $E[X]$.

Exercise 2

Consider a switch, whose output link has a capacity of C bits per second, connected to 50 input terminals. Each terminal sends packets whose length is exponentially distributed, with a mean of 1000 bits. The interarrival times of packets *from a terminal* are exponentially distributed. For half of the terminals, the mean interarrival time is 10 seconds. For the other half, it is 5 seconds.

1) Compute the value of C so that the mean number of packets in the switch is equal to 5
2) Compute the value of C so that the 99th percentile of the number of packets in the switch is equal to 5, call it C'.
3) Assume (for this point only) that the length of packets is *constant*. Solve again point 1). Comment on your findings.
4) Compute the 99th percentile of the response time of a packet as a function of C.

Exercise 1 – Solution

1) In order to be a CDF, the following conditions should be verified:
   a) $F(x)$ must be monotonic
   b) $lim_{x\to-\infty} F\,(x) = 0$
   c) $lim_{x\to+\infty} F\,(x) = 1$

b) always holds. c) holds if and only if $\alpha = 1$.
As far as monotonicity is concerned, we observe that $F(x)$ is identically null for $x \leq -5$, and that its derivative is:

$$F'(x) = \begin{cases} \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

when $x > -5$. Therefore, monotonicity is guaranteed if $\beta \geq 5$.
The conditions requested by 1) are $\alpha = 1, \beta \geq 5$.

2) As per the computations above, it is:

$$f(x) = \begin{cases} 0 & x < -5 \\ \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

3) It is:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{-5}^{0} x \cdot \frac{\beta+5}{(\beta-x)^2}\,dx + \int_{0}^{+\infty} x \cdot \frac{\beta-5}{(\beta+x)^2}\,dx =$$
$$(\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}\,dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}\,dx$$

Now, the first integral is always finite (since its limits are), whereas the second may not be. After few algebraic passages, we obtain:

$$\int_{0}^{+\infty} \frac{x}{(\beta+x)^2}\,dx = \int_{0}^{+\infty} \frac{(\beta+x)-\beta}{(\beta+x)^2}\,dx = \left[\frac{\beta}{\beta+x} + log|\beta+x|\right]_{0}^{+\infty} = +\infty$$

Therefore, the expectation exists only if $\beta = 5$.

4) Assuming $\beta = 5$, the expectation is equal to:

$$E[X] = (\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}\,dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}\,dx$$
$$= 10 \cdot \int_{-5}^{0} \frac{x}{(5-x)^2}\,dx = 10 \cdot \left[\frac{-5}{x-5} + log|x-5|\right]_{-5}^{0}$$
$$= 5 - 10\,log(2)$$

Exercise 2 - Solution

The system can be modeled as an M/M/1 queueing system where:

- $\mu = \frac{C}{E[L]} = \frac{C}{1000}$
- $\lambda = 25 \cdot \frac{1}{10} + 25 \cdot \frac{1}{5} = 7.5$

1) The mean number of jobs in an M/M/1 is given by Kleinrock's formula, i.e. $E[N] = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$. Imposing $E[N] = 5$ and solving the latter for C yields $C = 9000$.

2) We also know that, in an M/M/1, SS probabilities are $p_k = \rho^k \cdot (1-\rho)$, hence we need to impose that:

$$\sum_{k=0}^{5} p_k = (1-\rho) \cdot \frac{1-\rho^6}{1-\rho} = 0.99$$

This yields $\rho^6 = 0.01$, i.e. $\rho = \frac{1}{\sqrt[3]{10}} \cong 0.464$. Solving this equality for C yields

$$C' \cong 16163$$

3) If the length of the packets is constant, the system is an M/D/1, for which PK's formula yields the mean number of jobs in the system:

$$E[N] = \rho + \frac{\rho^2}{2(1-\rho)}$$

Imposing $E[N] = 5$ and solving the latter for ρ yields the following: $\rho = 6 \pm \sqrt{26}$. Since stability requires $\rho < 1$, the only good solution is $\rho = 6 - \sqrt{26} \cong 0.901$. Solving the latter for C yields $C \cong 8324$.
The value required is *smaller* than at point 1), since in this case there is no randomness in the transmission. Randomness increases queueing, as PK's formula shows.

4) In an M/M/1 system, the response time is an exponential RV, whose CDF is the following:

$$R(t) = 1 - e^{-\mu(1-\rho)t}$$

Hence the required equation is $1 - e^{-(\mu-\lambda)\pi_{99}} = 0.99$, to be solved for π_{99}. After a few straightforward computations, one gets:

$$\pi_{99} = \frac{2log10 \cdot E[L]}{C - 7.5 \cdot E[L]}$$