PECSN, 02/07/2025

Exercise 1

You have two independent unfair coins. The probability of *heads* are p and q, respectively, for coin 1 and 2. You play the following game: flip coin 1 until *heads* come up, and then flip coin 2 once. If coin 2 is *heads,* the game ends, otherwise you resume flipping coin 1, and so on.
Let T be the RV that counts the number of flips of coin 1 before the game ends.

1) Compute the PMF and the CDF of RV T.
2) Compute the mean value of T and its variance

Exercise 2

A network server handles *job lists*, which are sent from the outside world at a constant rate λ, with exponential interarrival times. A list includes an arbitrary number of *jobs*. The probability that a list includes n jobs is equal to π_n. The server processes individual jobs in FIFO order. A job's service time is an exponential RV with a mean equal to $\frac{1}{\mu}$. The server only accepts a new job list when it is idle.

1) model the system and draw the CTMC;
2) compute the steady-state probabilities and the stability condition; provide an interpretation for your findings.
3) compute the probability that a job list is rejected;
4) compute the *distribution* of the job's response time;
5) compute the server utilization.

Exercise 1 – Solution

The model is no different from the following: you *always* flip both coins simultaneously, and you win when you get two heads simultaneously. Since the coins are independent, the probability of getting two heads simultaneously is pq, hence T is a geometric RV with a probability of success pq (to be more specific, the version of a geometric RV that counts the number of *trials* before the first success). For the latter, we have:

$$p_k = P\{T = k\} = (1 - pq)^{k-1} \cdot pq$$
$$F(k) = P\{T \leq k\} = 1 - (1 - pq)^k$$
$$E[T] = \frac{1}{pq}$$
$$Var(T) = \frac{1 - pq}{(pq)^2}$$

To compute the mean and variance, one may resort to the PGF:

$$G(z) = \sum_{k=1}^{+\infty} z^k \cdot (1 - pq)^{k-1} \cdot pq = \frac{zpq}{1 - z + zpq}$$

And use the well-known relationships $E[T] = G'(1)$, $Var(T) = G"(1) + G'(1) - G'(1)^2$

If one misses the above trick, the PMF and CDF can still be found by taking an alternative (though considerably longer) route. Start from small values of k and compute the PMF and CDF manually:

- $k = 1$: $p_1 = pq$ and $F(1) = pq$
- $k = 2$:

$$p_2 = (1 - p)pq + p(1 - q)pq = pq(1 - pq)$$
$$F(2) = F(1) + p_2 = 1 - (1 - pq)^2$$

One can then observe the following general relationships:

$$p_k = P\{T = k | T > k - 1\} \cdot P\{T > k - 1\} = pq \cdot [1 - F(k - 1)]$$
$$F(k) = F(k - 1) + p_k = pq + (1 - pq) \cdot F(k - 1)$$

From which one gets:

$$p_3 = pq \cdot [1 - F(k - 1)] = pq \cdot (1 - pq)^2$$
$$F(3) = pq + (1 - pq) \cdot F(2) = 1 - (1 - pq)^3$$

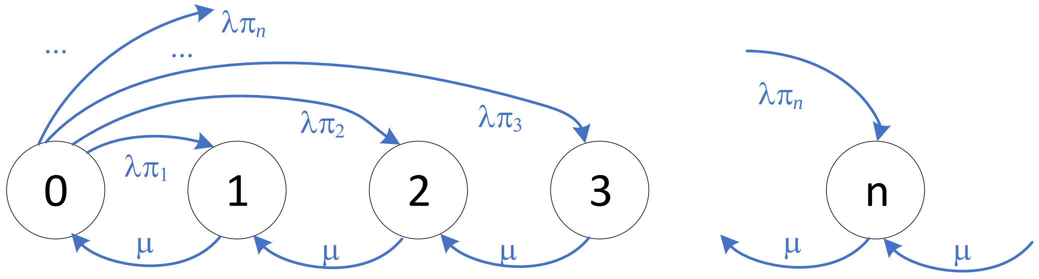
At this point, it is easy to venture that the general forms for the PMF and CDF should be:

$$p_k = pq \cdot (1 - pq)^{k-1}$$
$$F(k) = 1 - (1 - pq)^k$$

The above thesis can be proved by induction, exploiting the two general relationships above.

Exercise 2 - Solution

1) The CTMC is as follows. Normalization must hold in the π_n probabilities, i.e., $\sum_{n=1}^{+\infty} \pi_n = 1$.



2) The steady state probabilities are computed by writing down the global equilibrium equations:

$$p_0 \cdot \lambda = p_1 \cdot \mu$$
$$p_j \cdot \mu = p_{j+1} \cdot \mu + p_0 \cdot \lambda \cdot \pi_j, \ \ j \geq 1$$

From the above one readily obtains:

$$p_j = p_0 \cdot \frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i \, , \ \ j \geq 0$$

By imposing the normalization condition $\sum_{j=0}^{+\infty} p_j = 1$, one obtains the following:

$$p_0 + \sum_{j=1}^{+\infty} \left(p_0 \cdot \frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i \right) = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot \sum_{j=1}^{+\infty} \left(\sum_{i=j}^{+\infty} \pi_i \right) \right] = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot \sum_{j=1}^{+\infty} j \cdot \pi_j \right] = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot E[\Pi] \right] = 1$$
$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu} \cdot E[\Pi]}$$

The stability condition is that $E[\Pi]$ should be finite. Stability does not depend on λ or μ, since the server does not accept job lists unless idle, hence the rate of services and arrivals are immaterial. Moreover, we get:

$$p_j = \frac{\frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i}{1 + \frac{\lambda}{\mu} \cdot E[\Pi]} = \frac{1 - F_\Pi(j-1)}{\frac{\mu}{\lambda} + E[\Pi]}, \ \ j \geq 1$$

3) The probability that a job list is rejected is:

$$p_L = 1 - p_0 = \frac{E[\Pi]}{\frac{\mu}{\lambda} + E[\Pi]}$$

4) Whenever a list arrives, the system is empty by hypothesis. Therefore, the response time of a job arriving in a list that has $n-1$ jobs ahead of it will be an Erlang with n stages. The probability that an arriving job is the n^{th}in a list is:

$$\sum_{j=n}^{+\infty} \pi_j \cdot \frac{1}{j}$$

Since i) that list must include at least n jobs, and ii) that job must be the n^{th}in that list (while it could be any other with the same probability). Therefore, the required answer is:

$$P\{R \leq t\} = \sum_{n=1}^{+\infty} P\{R \leq t | n^{th}\, jobs\} \cdot P\{n^{th}\, job\} =$$

$$\sum_{n=1}^{+\infty} \left\{ \left[1 - \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \right] \cdot \left[\sum_{j=n}^{+\infty} \pi_j \cdot \frac{1}{j} \right] \right\}$$

5) The server utilization is equal to the loss probability:

$$U = p_L = 1 - p_0 = \frac{E[\Pi]}{\frac{\mu}{\lambda} + E[\Pi]}$$