**Exercise 1**

A sender sends a message to a receiver, consisting of a sequence of 10 independent bits. The sender uses randomly either device $d_1$ or device $d_2$ to send its message. Device $d_j$ transmits a bit equal to 1 with a probability $p_j$. The receiver decodes the message correctly if the sequence has *at least* eight 1s in it.

1) Compute the probability that the receiver decodes the message correctly;
2) Compute the probability that a message which is received correctly has been sent using device $d_1$;

Assume now that a message is received correctly if and only if it has all its bits to 1. Call $x = p_2/p_1$.

3) Draw the graph of the probability computed at point 2 – as correctly as you can – as a function of $x$;
4) Determine for which value of $x$ that probability becomes equal to 0.01.

**Exercise 2**

$N$ robots operate on a factory floor, coordinated by a central entity. Each robot performs a task, and then waits for a new order by the central entity. The time it takes for a robot to complete its task is an exponential random variable with a mean equal to $1/\lambda$. The central entity waits for *all* the robots to complete their assigned task, and then prepares a new task for all the robots to complete. The think time of the central entity is an exponential random variable with a mean equal to $1/\mu$. After that time, all robots are activated simultaneously.

1) Model the system and draw its CTMC
2) Find the steady-state probabilities and the stability condition.
3) Compute the utilization of the central entity.
4) Compute the mean time for which a robot is blocked waiting for a new order from the central entity.
5) Discuss what happens to the answers of points 3 and 4 in limit cases $\lambda \to \infty, \lambda \to 0$

**Exercise 1 – solution**

1) The probability of event $E_k \equiv \{k\ bits\ to\ 1\ in\ a\ message\}$, if transmitted by device $d_j$ is:

$$P\{E_k|d_j\} = \binom{10}{k} p_j^k \cdot (1-p_j)^{10-k}$$

The probability of having $k$ bits to 1 in a message is therefore:

$$P\{E_k\} = P\{E_k|d_1\} \cdot P\{d_1\} + P\{E_k|d_2\} \cdot P\{d_2\} = \frac{1}{2}(P\{E_k|d_1\} + P\{E_k|d_2\})$$

The message is decoded correctly if 8 or more bits are equal to 1, i.e.

$$P\{correct\} = \sum_{k=8}^{10} P\{E_k\} = \frac{1}{2}\sum_{k=8}^{10}\left[\binom{10}{k} p_1^k \cdot (1-p_1)^{10-k} + \binom{10}{k} p_2^k \cdot (1-p_2)^{10-k}\right]$$

2) By Bayes' theorem, we have:

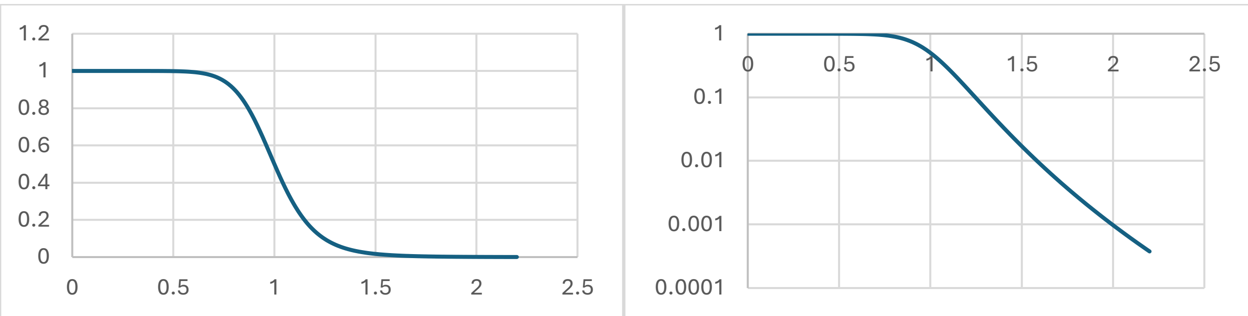
$$P\{d_1|correct\} = \frac{P\{correct|d_1\} \cdot P\{d_1\}}{P\{correct\}}$$

$$= \frac{\sum_{k=8}^{10}\left[\binom{10}{k} p_1^k \cdot (1-p_1)^{10-k}\right]}{\sum_{k=8}^{10}\left[\binom{10}{k} p_1^k \cdot (1-p_1)^{10-k} + \binom{10}{k} p_2^k \cdot (1-p_2)^{10-k}\right]}$$

3) The above formula simplifies to:

$$P\{d_1|correct\} = \frac{1}{1+\left(\frac{p_2}{p_1}\right)^{10}} = \frac{1}{1+x^{10}}$$

The above relation is plotted with the ordinates represented in both linear and log scales.




4) The equation to be solved in $x$ is

$$y = \frac{1}{1+x^{10}}$$

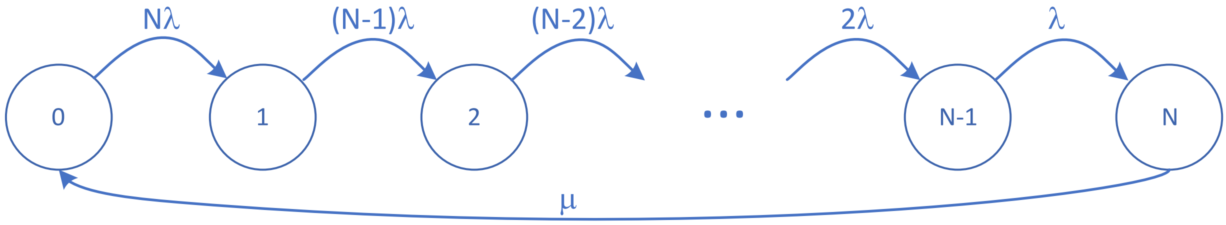
With $y = 0.01$. This yields

$$y = \frac{1}{1+x^{10}}$$

$$x = \sqrt[10]{99} \cong 1.58$$

**Exercise 2 – solution**

This system is a finite-population one. We use the number of waiting robots as its state characterization. The resulting CTMC is as follows.




The system has a finite number of states, hence it is always stable. The global SS equations are:

$$N\lambda p_0 = \mu p_N$$

$$(N-i)\lambda p_i = (N-i+1)\lambda p_{i-1}, \quad 1 \le i \le N-1$$

$$\lambda p_{N-1} = \mu p_N$$

From the above, we obtain:

$$p_i = \frac{N}{N-i} p_0, \quad 0 \le i \le N-1$$

$$p_N = N\frac{\lambda}{\mu} p_0$$

Hence normalization reads:

$$\sum_{i=0}^{N-1} \frac{N}{N-i} p_0 + N\frac{\lambda}{\mu} p_0 = 1$$

$$N p_0 \left(H_N + \frac{\lambda}{\mu}\right) = 1$$

Where $H_N$ is the $N$-th harmonic number. We get:

$$p_0 = \frac{1}{N\left(H_N + \frac{\lambda}{\mu}\right)}$$

$$p_i = \frac{1}{(N-i)\left(H_N + \frac{\lambda}{\mu}\right)}, \quad 1 \le i \le N-1$$

$$p_N = \frac{\frac{\lambda}{\mu}}{\left(H_N + \frac{\lambda}{\mu}\right)} = \frac{1}{\frac{\mu}{\lambda} H_N + 1}$$

The central entity is idle in states $0..N-1$. Its utilization is therefore equal to $p_N$. When $\lambda \to \infty$, the utilization approaches one (robots are extremely fast to complete their tasks, hence the central entity is always busy). When $\lambda \to 0$, the utilization drops to zero, because it takes forever before all robots complete their tasks.

The mean number of blocked robots can be computed as:

$$E[N] = \sum_{i=0}^{N} i \cdot p_i = \sum_{i=0}^{N-1} i \cdot \frac{1}{(N-i)\left(H_N + \frac{\lambda}{\mu}\right)} + Np_N =$$

$$\sum_{i=0}^{N-1} \frac{N-(N-i)}{(N-i)\left(H_N + \frac{\lambda}{\mu}\right)} + Np_N =$$

$$N\sum_{i=0}^{N-1} \frac{1}{(N-i)\left(H_N + \frac{\lambda}{\mu}\right)} - \sum_{i=0}^{N-1} \frac{1}{H_N + \frac{\lambda}{\mu}} + Np_N =$$

$$N(1-p_N) - \sum_{i=0}^{N-1} \frac{1}{H_N + \frac{\lambda}{\mu}} + Np_N =$$

$$N\left(1 - \frac{1}{H_N + \frac{\lambda}{\mu}}\right)$$

The mean arrival rate is:

$$\gamma = \bar{\lambda} = \sum_{i=0}^{N-1} (N-i)\lambda p_i = \frac{N\lambda}{H_N + \frac{\lambda}{\mu}}$$

Hence we get:

$$E[R] = \frac{E[N]}{\gamma} = N\frac{H_N + \frac{\lambda}{\mu} - 1}{H_N + \frac{\lambda}{\mu}} \cdot \frac{H_N + \frac{\lambda}{\mu}}{N\lambda} =$$

$$\frac{H_N - 1}{\lambda} + \frac{1}{\mu}$$

When $\lambda \to \infty$, we get $E[R] \to 1/\mu$. This makes perfect sense, since the decision time of the central entity is a lower bound to the blocking time of robots.

When $\lambda \to 0$, we get $E[R] \to +\infty$, because robots that complete a task need to wait forever before being unblocked again by the central entity.