Università di Pisa
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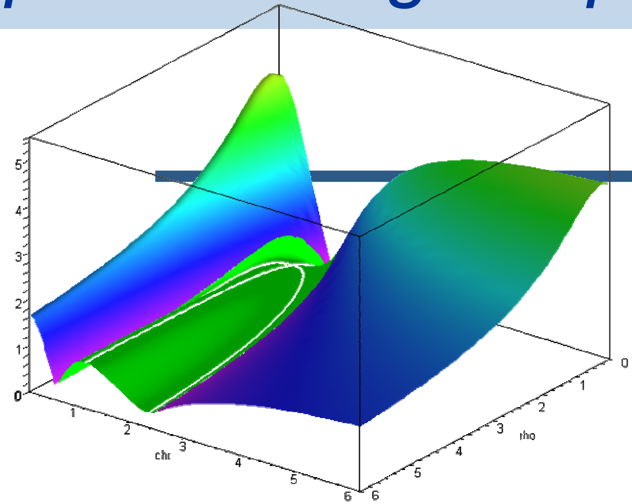
Influence of anisotropy on flexural optimal design of plates and laminates



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Foreword

The use of composite materials forces designers to use optimal procedures for obtaining non intuitive suitable solutions.

Designing of laminates with respect to flexural properties is the most cumbersome task in the design of laminates; few researches have been carried on in this field, and the most part of them lead to only approximate solutions.

The first task of this research was to find exact optimal solutions to some classical flexural problems of plates, when such plates are laminates composed of anisotropic identical plies.

A second task was that of assessing the influence of the anisotropy of the material on the optimal solutions so found: this is the topic of this talk.

Dimensionless invariant material properties have been chosen to represent the layer elastic properties, along with other dimensionless parameters describing geometry, deformation and/or loading.

Some unattended features and *pathological* cases have been so found.

Content

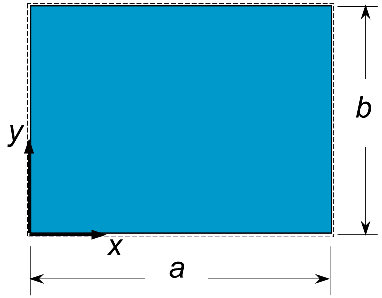
- Governing equations
- Dimensionless design parameters
- A common problem for bending, buckling and vibrations
- Analysis of the objective function
- Effectiveness of the optimal solution
- Bounds on optimal and anti-optimal solutions
- The case of a non-sinusoidal load
- The optimal critical load of buckling
- The optimal fundamental frequency
- Some examples of exact optimal solutions

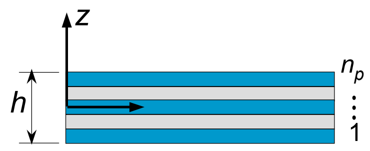
Governing equations: *the mechanical model*

Simply supported rectangular laminate made of identical layers.

Classical lamination theory (Kirchhoff model etc.).

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^o \\ \boldsymbol{\kappa} \end{Bmatrix}$$





Bending stiffness tensor:

$$\mathbf{D} = \frac{1}{12}\frac{h^3}{n_p^3}\sum_{j=1}^{n_p} d_j \mathbf{Q}(\delta_j), \qquad \mathbf{D} = \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{xy} & D_{yy} & D_{ys} \\ D_{xs} & D_{ys} & D_{ss} \end{bmatrix}.$$

$$d_j = 12j(j - n_p - 1) + 4 + 3n_p(n_p + 2) \quad \text{and} \quad \sum_{j=1}^{n_p} d_j = n_p^3.$$

Governing equations: *supplementary assumptions*

In order to dispose of an analytical solution, the laminate is assumed to be *specially orthotropic* in bending:

$$\mathbf{B}=\mathbf{O}; \quad D_{xs}=D_{ys}=0.$$

In this way, the equilibrium equation for deflection w is not coupled to the equations of in-plane displacements and the separation of variables is possible: the Navier's method can be applied.

For buckling, a further assumption is that

$$\mathbf{N}=(N_x, N_y, 0)$$

i.e. no shearing in-plane forces (if not, the Navier's method does not apply).

Governing equations: *transverse equilibrium equation*

$$[L_{33}](w) = p_z \quad \rightarrow \quad \text{equilibrium equation;}$$

$$[L_{33} - L_\lambda](w) = 0 \quad \rightarrow \quad \text{buckling equation;}$$

$$[L_{33} - L_\omega](w) = 0 \quad \rightarrow \quad \text{vibration equation;}$$

$$L_{33} = D_{xx}\frac{\partial^4}{\partial x^4} + 2(D_{xy} + 2D_{ss})\frac{\partial^4}{\partial x^2 \partial y^2} + D_{yy}\frac{\partial^4}{\partial y^4}.$$

$$L_\lambda = N_x\frac{\partial^2}{\partial x^2} + 2N_s\frac{\partial^2}{\partial x \partial y} + N_y\frac{\partial^2}{\partial y^2}, \quad L_\omega = \mu\frac{\partial^2}{\partial t^2},$$

Governing equations: *Navier's solution method*

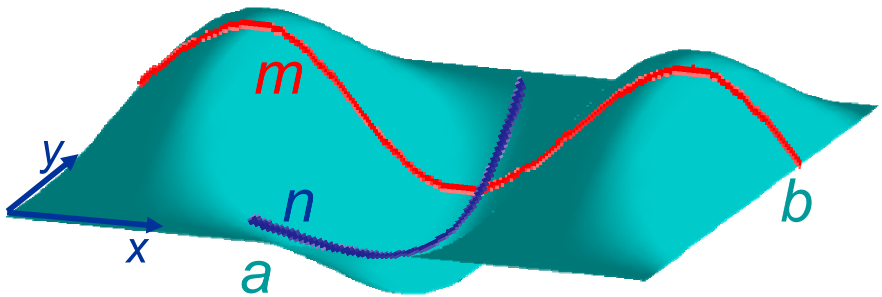
$$p_z(x,y) = \sum_{m,n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

$$w(x,y) = \sum_{m,n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} [\sin \omega_{mn} t].$$

$$p_{mn} = \psi \frac{P}{ab} p^{*}_{mn}, \quad \text{with}$$

$$\psi = \frac{16}{\pi^2} \text{ and } p^{*}_{mn} = \frac{1}{mn} \quad \text{for a uniform load } p = \frac{P}{ab},$$

$$\psi = 4 \text{ and } p^{*}_{mn} = \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \quad \text{for a concentrat ed load } P \text{ in the center.}$$



Governing equations: *polar parameters of the material*

$$\begin{aligned}
T_{xx} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1,\\
T_{xs} &= R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
T_{xy} &= -T_0 + 2T_1 - R_0\cos 4\Phi_0,\\
T_{ss} &= T_0 - R_0\cos 4\Phi_0,\\
T_{ys} &= -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
T_{yy} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1.
\end{aligned}$$

$\Phi_0 - \Phi_1 = k\,\pi/4$, k=0, 1: common orthotropy

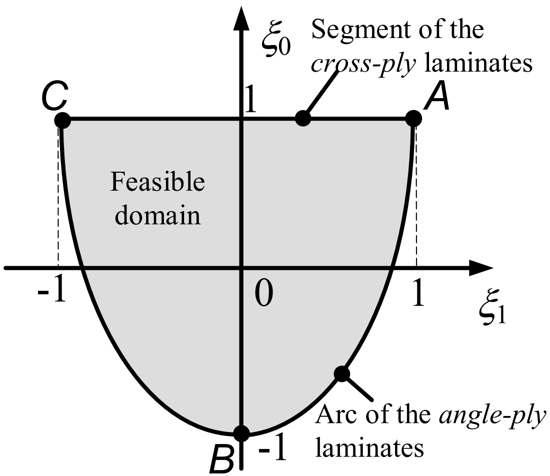
R_0=0: R_0-orthotropy

R_1=0: square symmetric orthotropy

Governing equations: *laminate bending stiffness*

Separation of *geometry* (lamination parameters) and *material* (polar constants):

$$\begin{Bmatrix} D_{xx} \\ D_{yy} \\ D_{xy} \\ D_{ss} \\ D_{xs} \\ D_{ys} \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} 1 & 2 & \xi_0 & 4\xi_1 \\ 1 & 2 & \xi_0 & -4\xi_1 \\ -1 & 2 & -\xi_0 & 0 \\ 1 & 0 & -\xi_0 & 0 \\ 0 & 0 & \xi_2 & 2\xi_3 \\ 0 & 0 & -\xi_2 & 2\xi_3 \end{bmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ (-1)^k R_0 \\ R_1 \end{Bmatrix}.$$



$$\xi_0 = \frac{1}{n_p^3}\sum_{j=1}^{n_p} d_j \cos 4\delta_j, \quad \xi_1 = \frac{1}{n_p^3}\sum_{j=1}^{n_p} d_j \cos 2\delta_j,$$

$$\xi_2 = \frac{1}{n_p^3}\sum_{j=1}^{n_p} d_j \sin 4\delta_j, \quad \xi_3 = \frac{1}{n_p^3}\sum_{j=1}^{n_p} d_j \sin 2\delta_j.$$

$$\boxed{-1 \le \xi_1 \le 1, \quad 2\xi_1^2 - 1 \le \xi_0 \le 1.}$$

Specially orthotropic laminates: ξ_0 and ξ_1 are sufficient to completely describe bending stiffness.

Governing equations: *polar form of the equations*

Separation of the mean isotropic part from the pure anisotropic part:

$$\frac{h^3}{12}(T_0+2T_1)\Delta\Delta w+\left[\tilde{L}_{33}(-L_\lambda-L_\omega)\right](w)=p_z,$$

$$\tilde{L}_{33}=\frac{h^3}{12}\left\{\left[(-1)^k R_0\xi_0+4R_1\xi_1\right]\frac{\partial^4}{\partial x^4}-6(-1)^k R_0\xi_0\frac{\partial^4}{\partial x^2\partial y^2}+\left[(-1)^k R_0\xi_0-4R_1\xi_1\right]\frac{\partial^4}{\partial y^4}\right\}$$

Mean isotropic part: meaning of the polar isotropy constants

$$T_0+2T_1=\frac{E}{1-\nu^2}.$$

Dimensionless parameters: *material*

$$\rho = \frac{R_0}{R_1}, \quad \tau = \frac{T_0 + 2T_1}{\sqrt{R_0^2 + R_1^2}}, \quad k = 4\frac{\Phi_0 - \Phi_1}{\pi}.$$

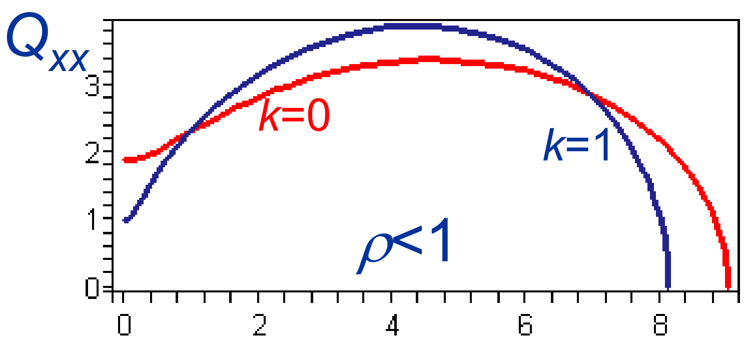
ρ : *anisotropy ratio:* ρ =0, R_0-orthotropic materials;

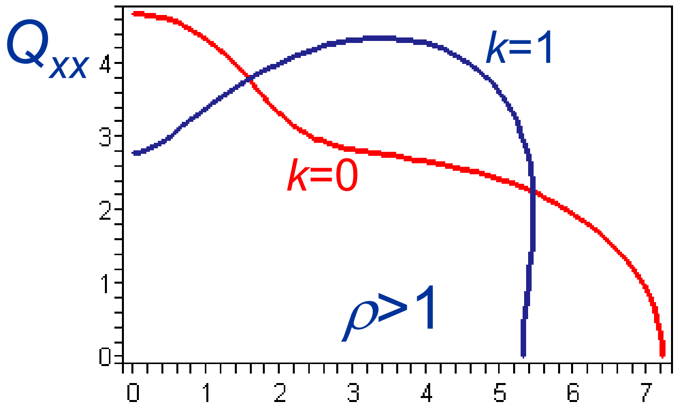
ρ =∞ : square-symmetric materials.

τ : *isotropy-to-anisotropy* ratio; $\tau > 1$

k: *orthotropy index*; *k*=0 : *low shear modulus orthotropy*

k=1 : *high shear modulus orthotropy*





Dimensionless parameters: *Cartesian expression*

For a generic orthotropic elasticity tensor **T** it is:

$$\rho = \frac{T_{xx} + T_{yy} - 2(T_{xy} + 2T_{ss})}{T_{xx} - T_{yy}},$$

$$\tau = \frac{3(T_{xx} + T_{yy}) + 2(T_{xy} + 2T_{ss})}{\sqrt{4(T_{xy} + 2T_{ss})(T_{xy} + 2T_{ss} - T_{xx} - T_{yy}) + 2(T_{xx}^2 + T_{yy}^2)}}.$$

$$k = 0 \quad \text{if} \quad T_{xx} + T_{yy} > 2(T_{xy} + 2T_{ss}),$$

$$k = 1 \quad \text{if} \quad T_{xx} + T_{yy} < 2(T_{xy} + 2T_{ss}).$$

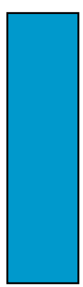
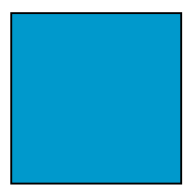
Dimensionless parameters: *geometry and mode*

Aspect ratio: $$\eta = \frac{a}{b}.$$

Mode ratio: $$\gamma = \frac{m}{n}.$$

Wave-length ratio: $$\chi = \frac{\eta}{\gamma} = \frac{na}{mb}.$$

Force ratio: $$\nu = \frac{N_y}{N_x}.$$

If m=n, χ=0 $\rightarrow$  χ=1 $\rightarrow$  χ=∞ $\rightarrow$ 

A common problem for bending, buckling and vibrations: *bending stiffness*

Maximization of the bending stiffness= minimization of the compliance $J_{\mathbf{D}}$

$$J_{\mathbf{D}} = \int_0^a \int_0^b p_z w \, dx \, dy.$$

Navier's solution of equilibrium equation

$$a_{mn} = \frac{1}{\pi^4} \frac{p_{mn}}{D_{xx}\alpha^2 + 2(D_{xy} + 2D_{ss})\alpha\beta + D_{yy}\beta^2}, \qquad \alpha = \frac{m^2}{a^2}, \beta = \frac{n^2}{b^2}.$$

Replacing the D_{ij} by their polar expressions we get

$$J_{\mathbf{D}} = \frac{3ab}{\pi^4 h^3} \sum_{m,n=1}^{\infty} \frac{p_{mn}^2}{(T_0 + 2T_1)(\alpha + \beta)^2 + (-1)^k R_0 \xi_0 (\alpha^2 + \beta^2 - 6\alpha\beta) + 4R_1 \xi_1 (\alpha^2 - \beta^2)}.$$

A common problem for…: *bending stiffness*

Using the dimensionless parameters above, we get

$$J_{\mathbf{D}} = \frac{3\psi^2\eta}{\pi^4}\frac{P^2a^2}{h^3\sqrt{R_0^2+R_1^2}}\sum_{m,n=1}^{\infty}\frac{p_{mn}^{*2}}{m^4(1+\chi^2)^2\varphi(\xi_0,\xi_1)},$$

with

$$\varphi(\xi_0,\xi_1) = \tau + \frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k\rho\xi_0\frac{\chi^4-6\chi^2+1}{(1+\chi^2)^2}+4\xi_1\frac{1-\chi^2}{1+\chi^2}\right].$$

Further simplification: for a material, geometry and a sinusoidal load given (i.e. for fixed m and n), the optimization problem is reduced to the maximization of the function $\varphi(\xi_0,\ \xi_1)$.

A common problem for…: *buckling loads*

Be **N**=λ *(*N_x*,* N_y*, 0),* λ = load multiplier.

We want to maximise λ_{mn}, the buckling load multiplier for the mode (m, n).

The Navier's non-trivial solution of the buckling equation for the mode (m, n) is

$$\lambda_{mn} = \pi^2 \frac{D_{xx}\alpha^2 + 2(D_{xy} + 2D_{ss})\alpha\beta + D_{yy}\beta^2}{N_x\alpha + N_y\beta}.$$

Once again, replacing the D_{ij} by their polar expressions and using the dimensionless parameters above, we get

$$\lambda_{mn} = \frac{\pi^2 m^2 h^3}{12a^2}\sqrt{\frac{R_0^2 + R_1^2}{N_x^2 + N_y^2}}(1+\chi^2)^2 \frac{\sqrt{1+\nu^2}}{1+\nu\,\chi^2}\varphi(\xi_0,\xi_1).$$

A common problem for…: *natural frequencies*

Finally, we consider the problem of maximising the natural frequency ω_{mn} of a given mode (m, n).

The Navier's non-trivial solution of the vibration equation for the mode (m, n) is

$$\omega_{mn}^2 = \frac{\pi^4}{\mu}\left[D_{xx}\alpha^2 + 2(D_{xy} + 2D_{ss})\alpha\beta + D_{yy}\beta^2\right].$$

μ: mass of the laminate per unit area of the plate's surface.

As usual, replacing the D_{ij} by their polar expressions and using the dimensionless parameters above, we get

$$\omega_{mn}^2 = \frac{\pi^4 m^4 h^3}{12\mu\, a^4}\sqrt{R_0^2 + R_1^2}(1+\chi^2)^2\varphi(\xi_0,\xi_1).$$

A common problem for…: *the objective function*

Finally, the three problems above, concerning the flexural behaviour of the laminate for a precise mode, are reduced to the same non linear optimization problem:

$$\text{maximise} \quad \varphi(\xi_0, \xi_1),$$
$$\text{subjected to} \quad -1 \le \xi_1 \le 1, \quad 2\xi_1^2 - 1 \le \xi_0 \le 1.$$

To remark that also the opposite problem (that we will call the *anti-optimization* one) is physically meaningful, as the objective function can be proved to be always positive:

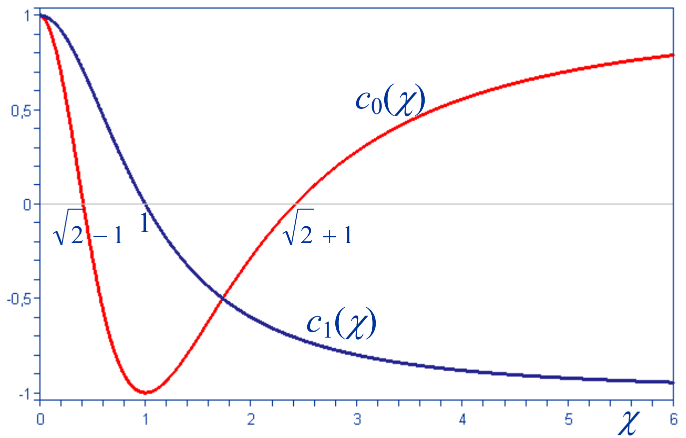
$$\text{minimise} \quad \varphi(\xi_0, \xi_1),$$
$$\text{subjected to} \quad -1 \le \xi_1 \le 1, \quad 2\xi_1^2 - 1 \le \xi_0 \le 1.$$

Analysis of the objective function: *separation of material and mode*

We rewrite the objective function φ as

$$\varphi(\xi_0,\xi_1)=\tau+\frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k\rho\, c_0(\chi)\xi_0+4\,c_1(\chi)\xi_1\right],$$

with $c_0(\chi)=\dfrac{\chi^4-6\chi^2+1}{(1+\chi^2)^2},\quad c_1(\chi)=\dfrac{1-\chi^2}{1+\chi^2}.$



The functions $c_0(\chi)$ and $c_1(\chi)$ give the influence of the mode and geometry. Their roots are of some importance.

ρ, τ and k give the influence of the material

τ : gives the influence of the ply's isotropy

ρ : gives the influence of the ply's anisotropy

k : gives the influence of the orthotropy's type

Analysis of the objective function: *pathological solutions*

$\varphi(\xi_0,\xi_1)$ is linear with respect to ξ_0, ξ_1 $\rightarrow$ maxima and minima are located on the boundary of the feasible domain.

Nonetheless, it is useful to analyse the gradient of $\varphi(\xi_0,\xi_1)$:

$$\nabla\varphi(\xi_0,\xi_1)=\left(\frac{\partial\varphi}{\partial\xi_0},\frac{\partial\varphi}{\partial\xi_1}\right)=\left(\frac{(-1)^k\rho}{\sqrt{1+\rho^2}}c_0(\chi),\frac{4}{\sqrt{1+\rho^2}}c_1(\chi)\right),$$

$\nabla\varphi(\xi_0,\xi_1)=0 \leftrightarrow$

> $\rho=0$ and $\chi=1$: this is the case of laminates made of R_0 -orthotropic materials ($R_0=0$) and with equal wave-length of the mode along x and y, (*e.g.* square plates and modes with $m=n$);

or

Analysis of the objective function: *pathological solutions*

$\rho=\infty$ and $\chi=\sqrt{2}\pm1$: this is the case of laminates made of square-symmetric materials (R_1=0), *i.e.* reinforced by balanced fabrics, and, if for instance *m*=*n*, having an aspect ratio $\eta=\sqrt{2}\pm1$.

In these two circumstances, it is not possible to optimize the laminate, because the objective function is constant and reduces to only its isotropic part, τ.

Actually, in such cases, the contribution of the anisotropic part disappears, due to special combinations of geometry, mode and anisotropy properties of the layer: *the laminate behaves like it was made of isotropic layers*, and any possible stacking sequence give the same result.

Analysis of the objective function: *cross-ply solutions*

Cross-ply laminates are represented by lamination points of the type ξ_0=1, $-1\leq \xi_1\leq 1$; this is possible $\leftrightarrow$

$$\nabla\varphi(\xi_0,\xi_1)=(a^2,0),\quad a\in\mathbb{R}-\{0\}.$$

$$(-1)^k c_0(\chi)>0 \quad\Leftrightarrow\quad \begin{cases} k=0 \quad \text{and} \quad \chi\in[0,\sqrt{2}-1) \ \text{ or } \ \chi>\sqrt{2}+1, \\ \qquad k=1 \quad \text{and} \quad \sqrt{2}-1<\chi<\sqrt{2}+1; \end{cases}$$

$$\frac{c_1(\chi)}{\sqrt{1+\rho^2}}=0 \quad\Leftrightarrow\quad \chi=1 \quad \text{or} \quad \rho=\infty.$$

For the anti-optimal problem, it is sufficient in the first condition above to change k=0 into k=1 and *vice-versa*, *i.e.*, k changes maxima into minima and *vice-versa*: this is typical.

Analysis of the objective function: *cross-ply solutions*

A remark: cross-ply solutions exist only in the presence of a *generalised square-symmetry*: of the material, condition $\rho=\infty$, or of the geometry and mode, condition $\chi=1$ (*e.g.*, $m=n$ and a square plate).

The values of the solutions are

$$\text{for } \chi=1,\ \varphi=\tau-\frac{(-1)^k\rho}{\sqrt{1+\rho^2}},$$

$$\text{for } \rho=\infty,\ \varphi=\tau+(-1)^k c_0(\chi).$$

To notice that in the first case ρ influences the extreme values, while χ in the second:

Analysis of the objective function: *cross-ply solutions*

$$\varphi_{max} = \max\left[\tau - \frac{(-1)^k \rho}{\sqrt{1+\rho^2}}\right]_{k=1,\rho=\infty} = \max\left[\tau + (-1)^k c_0(\chi)\right]_{\substack{(k=0,\chi=\{0,\infty\})\\ \text{or } (k=1,\chi=1)}} = \tau + 1,$$

$$\varphi_{min} = \min\left[\tau - \frac{(-1)^k \rho}{\sqrt{1+\rho^2}}\right]_{k=0,\rho=\infty} = \min\left[\tau + (-1)^k c_0(\chi)\right]_{\substack{(k=1,\chi=\{0,\infty\})\\ \text{or } (k=0,\chi=1)}} = \tau - 1.$$

Optimal and anti-optimal cross-ply solutions are not unique, as ξ_1 disappears from the different expressions above: any laminate combination of layers at 0° and at 90° is an optimal (or anti-optimal) solution if conditions above are satisfied.

Analysis of the objective function: *angle-ply solutions*

Angle-ply laminates are located on the boundary $\xi_0 = 2\xi_1^2 - 1$ of the feasible domain, where

$$\varphi(\xi_1) = \tau + \frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k \rho\, c_0(\chi)(2\xi_1^2 - 1) + 4\, c_1(\chi)\xi_1\right],$$

whose maxima and minima can be only

$$\varphi_{11} = \varphi(\xi_1 = 1) = \tau + \frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k \rho\, c_0(\chi) + 4\, c_1(\chi)\right],$$

$$\varphi_{22} = \varphi(\xi_1 = -1) = \tau + \frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k \rho\, c_0(\chi) - 4\, c_1(\chi)\right],$$

$$\varphi_{\delta\delta} = \varphi(\xi_1 = \xi_1^{st}) = \tau - \frac{(-1)^k}{\sqrt{1+\rho^2}} \frac{\rho^2 c_0^2(\chi) + 2 c_1^2(\chi)}{\rho\, c_0(\chi)},$$

Analysis of the objective function: *angle-ply solutions*

The corresponding orientation angles δ are

$$\xi_1 = 1 \;\; \text{corresponds to} \;\; \delta = 0, \quad \rightarrow \quad \text{unidirectional}$$

$$\xi_1 = -1 \;\; \text{correponds to} \;\; \delta = \frac{\pi}{2}, \quad \rightarrow \quad \text{unidirectional}$$

$$\xi_1 = \xi_1^{st} \;\; \text{correponds to} \;\; \delta = \frac{1}{2}\arccos \xi_1^{st} \quad \rightarrow \quad \text{true angle} - \text{ply}$$

with

$$\left.\frac{\partial \varphi}{\partial \xi_1}\right|_{\xi_1^{st}} = 0 \quad \Rightarrow \quad \xi_1^{st} = -\frac{(-1)^k}{\rho}\frac{c_1(\chi)}{c_0(\chi)},$$

Remark: it is easy to verify that for two plates having reciprocal wave-length ratios, the respective solution angles δ are complementary.

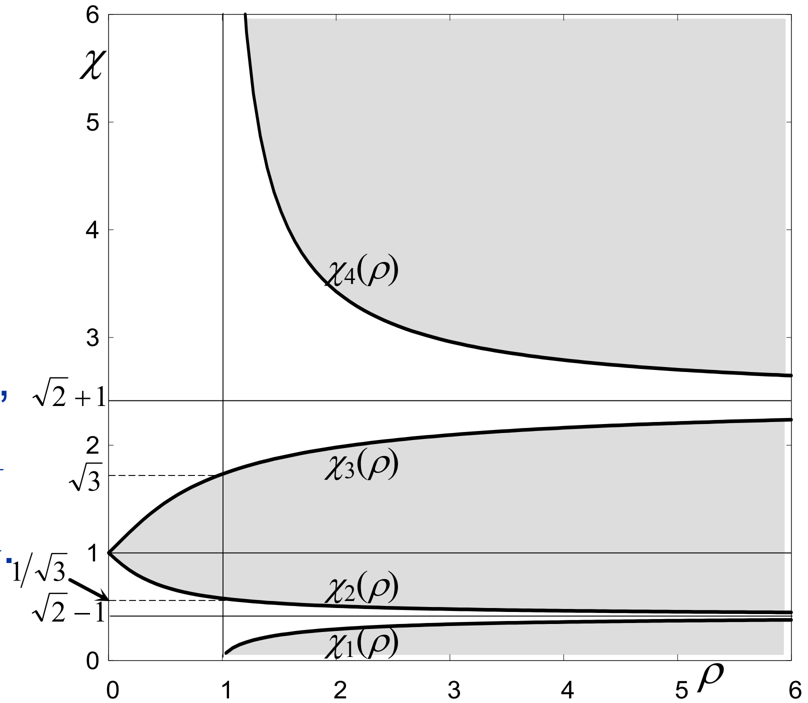
Analysis of the objective function: *angle-ply solutions*

Angle-ply solutions exist $\leftrightarrow$ $-1 \le \dfrac{(-1)^k}{\rho}\dfrac{c_1(\chi)}{c_0(\chi)} \le 1.$

This conditions give link the influence of the material part to that of the mode on the existence of angle-ply optimal laminates:

$$\rho > 1 \quad \text{and} \quad 0 < \chi \le \chi_1(\rho),$$
$$\rho \ge 0 \quad \text{and} \quad \chi_2(\rho) < \chi \le \chi_3(\rho),$$
$$\rho > 1 \quad \text{and} \quad \chi \ge \chi_4(\rho);$$

$$\chi_1(\rho) = \sqrt{\frac{3\rho - \sqrt{8\rho^2+1}}{\rho+1}}, \quad \chi_2(\rho) = \sqrt{\frac{3\rho - \sqrt{8\rho^2+1}}{\rho-1}},$$
$$\chi_3(\rho) = \sqrt{\frac{3\rho + \sqrt{8\rho^2+1}}{\rho+1}}, \quad \chi_4(\rho) = \sqrt{\frac{3\rho + \sqrt{8\rho^2+1}}{\rho-1}}.$$



Analysis of the objective function: *angle-ply solutions*

Once more *k* changes minima into maxima:

$$\frac{\partial^2 \varphi}{\partial \xi_1^2} = 4(-1)^k \frac{\rho}{\sqrt{1+\rho^2}} c_0(\chi).$$

In particular, using the expression of $c_0(\chi)$ we find that

- if $(\rho > 1, 0 < \chi \leq \chi_1(\rho))$ or $(\rho > 1, \chi \geq \chi_4(\rho))$, then $\varphi_{\delta\delta} = \varphi_{max}$ for $k = 1$, $\varphi_{\delta\delta} = \varphi_{min}$ for $k = 0$;
- if $(\rho \geq 0, \chi_2(\rho) < \chi \leq \chi_3(\rho))$, then $\varphi_{\delta\delta} = \varphi_{max}$ for $k = 0$, $\varphi_{\delta\delta} = \varphi_{min}$ for $k = 1$.

Remark: the isotropy parameter τ does not affect the solution.

Analysis of the objective function: *map of the solutions*

Considering the hierarchy of φ_{11}, φ_{22} and $\varphi_{\delta\delta}$ we see that

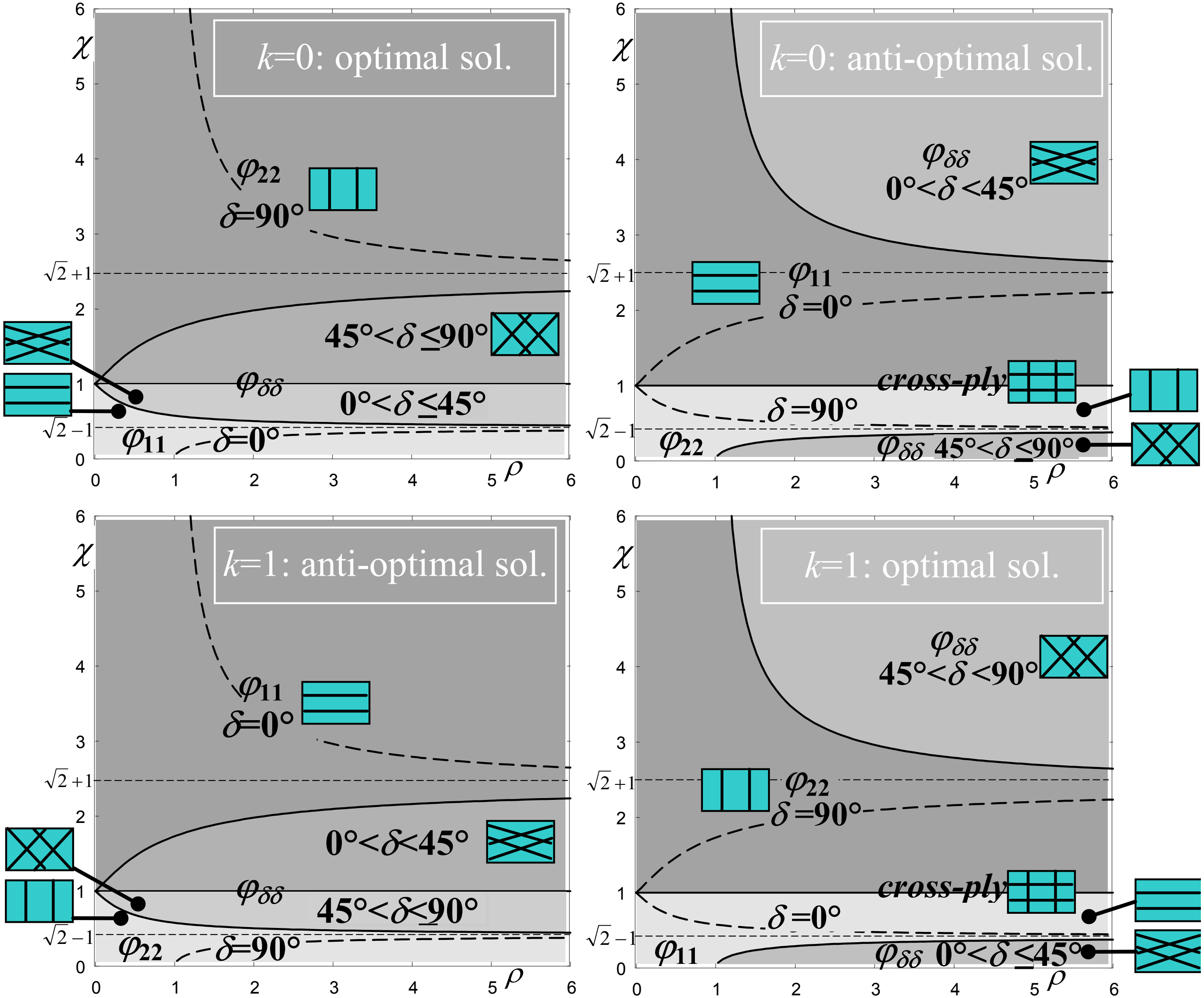
$$\varphi_{11} > \varphi_{22} \quad \Leftrightarrow \quad c_1(\chi) > 1 \quad \Leftrightarrow \quad \chi > 1;$$

$$\varphi_{\delta\delta} > \begin{matrix} \varphi_{11} \\ \varphi_{22} \end{matrix} \quad \Leftrightarrow \quad \begin{cases} (-1)^{k-1}\left[(-1)^k \rho\, c_0(\chi) \pm c_1(\chi)\right]^2 > 0 \\ c_0(\chi) > 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \sqrt{2}-1 < \chi < \sqrt{2}+1 & \text{if } k = 0, \\ 0 < \chi < \sqrt{2}-1 \text{ or } \chi > \sqrt{2}+1 & \text{if } k = 1. \end{cases}$$

Crossing all these results, we can trace a map of the optimal and anti-optimal solutions in the plane (ρ, χ):

Analysis of the objective function: *map of the solutions*

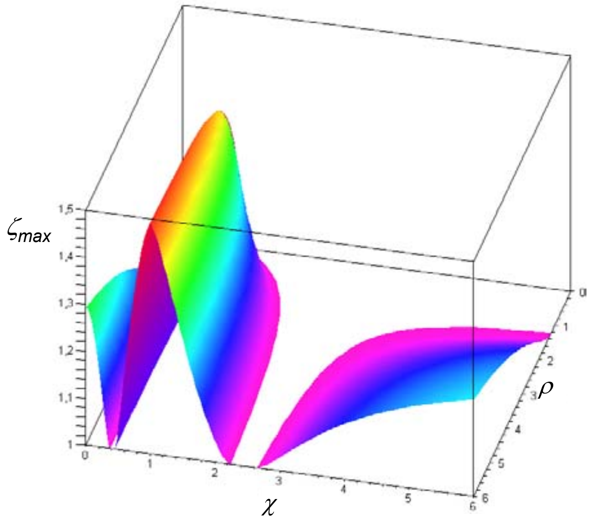


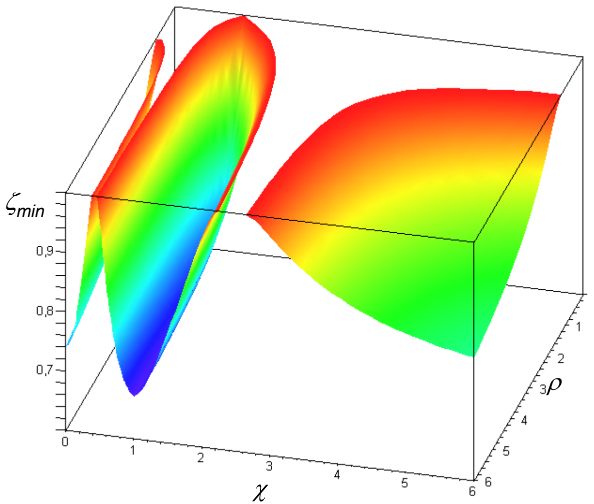
Effectiveness of the optimal solution

It is interesting to evaluate the gain of the *true* (angle-ply) optimal or anti-optimal solution with respect to the *intuitive* (unidirectional) one, *i.e.* the ratios

$$\zeta_{max}(\tau,\rho,\chi)=\frac{\varphi_{\delta\delta}}{\varphi_{ii}},\quad i=1\quad \text{if}\quad \chi<1,\quad i=2\quad \text{if}\quad \chi>1;$$

$$\zeta_{min}(\tau,\rho,\chi)=\frac{\varphi_{\delta\delta}}{\varphi_{ii}},\quad i=2\quad \text{if}\quad \chi<1,\quad i=1\quad \text{if}\quad \chi>1;$$





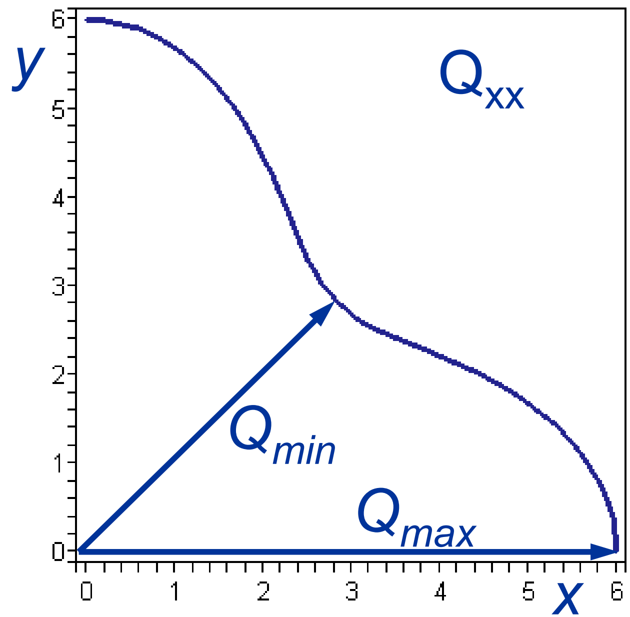
Effectiveness of the optimal solution: *extreme value*

These ratios get their extreme value for $\rho\rightarrow\infty$ and χ=0, χ=1 or $\chi\rightarrow\infty$, *i.e.*, if $m=n$, for square plates or infinite strips composed by square symmetric layers (R_1=0):

$$\max(\zeta_{max}) = \frac{\tau+1}{\tau-1} = \frac{Q_{max}}{Q_{min}},$$

$$\min(\zeta_{min}) = \frac{\tau-1}{\tau+1} = \frac{Q_{min}}{Q_{max}}.$$

The effectiveness of the solution is determined by the value of τ.



Effectiveness of the optimal solution: *real materials*

It can be shown that materials with k=1 are less diffused than those with k=0, but they do exist.

Apart from square symmetric layers, ρ =∞, the most part of composite layers have ρ<1 and k=0 → only the left part of the map of solutions is usually of concern.

If ρ<1, the range of χ where optimisation is meaningful, *i.e.* where the solution is not 0° or 90°, increases with ρ and, for ρ=1, it is comprised between $\chi = 1/\sqrt{3}$ and $\chi = \sqrt{3}$.

For current materials (ρ≈1, k=0), it is ζ_{max}=2.

Some examples of materials are in the following table:

Effectiveness of the optimal solution: *real materials*

Material	Fir wood	Ice	Titanium-Boride TiB_2	Boron-epoxy B(4)-55054	Carbon-epoxy T300-5208	Kevlar-epoxy 149	S-Glass-epoxy S2-449/SP 381	Glass-epoxy balanced fabric	Braided carbon-epoxy BR45a	Braided carbon-epoxy BR60
Reference	Lekhnitskii, 1950	Cazzani & Rovati, 2003	Cazzani & Rovati, 2003	Tsai & Hahn, 1980	Tsai & Hahn, 1980	Daniel & Ishai, 1994	MIL-HDBK-17-2F, 2002	Daniel & Ishai, 1994	Falzon & Herszberg, 1998	Falzon & Herszberg, 1998
E_1	10	11.75	387.60	205.00	181.00	86.90	47.66	29.70	40.40	30.90
E_2	0.42	9.61	253.81	18.50	10.30	5.52	13.31	29.70	19.60	42.60
G_{12}	0.75	3.00	250.00	5.59	7.17	2.14	4.75	5.30	25.00	14.00
ν_{12}	0.01	0.27	0.44	0.23	0.28	0.34	0.27	0.17	0.75	0.34
Q_{11}	10	12.51	445.62	206.00	181.81	87.54	48.65	30.58	55.56	36.76
Q_{22}	0.42	10.22	291.80	18.59	10.35	5.56	13.59	30.58	26.96	50.68
Q_{66}	0.75	3.00	250.00	5.59	7.17	2.14	4.75	5.30	25	14.00
Q_{12}	0.004	2.78	130.12	4.27	2.89	1.89	3.67	5.20	20.22	17.23
T_0	1.68	3.65	184.65	29.80	26.88	12.23	92.38	8.99	17.76	13.62
T_1	1.30	3.54	124.71	29.14	24.74	12.11	86.97	8.94	15.37	15.24
R_0	0.93	0.65	65.35	24.21	19.71	10.09	44.86	3.70	7.24	0.38
R_1	1.19	0.28	19.23	23.42	21.43	10.25	43.82	0	3.57	1.74
Φ_0	0	0	$\pi/4$	0	0	0	0	0	$\pi/4$	$\pi/4$
Φ_1	0	0	0	0	0	0	0	0	0	$\pi/2$
k	0	0	1	0	0	0	0	0	1	(–)1
ρ	0.78	2.32	3.40	1.03	0.92	0.98	1.02	∞	2.03	0.22
τ	2.83	15.16	6.37	2.61	2.62	2.53	4.25	7.26	6.01	24.76
V_f	-	-	-	0.50	0.70	0.60	0.50	0.45	0.60	0.60

(modules in GPa)

Bounds on optimal and anti-optimal solutions

We consider the influence of ρ and χ upon φ_{max} and φ_{min}, *i.e.* we look for the curves χ=$\chi(\rho)$ in the plane (ρ, χ) where the surfaces φ_{11}, φ_{22} and $\varphi_{\delta\delta}$ have a local or absolute maximum (minimum) with respect to χ:

$$\frac{\partial \varphi_{ii}}{\partial \chi} = 0, \quad i = 1,2,\delta;$$

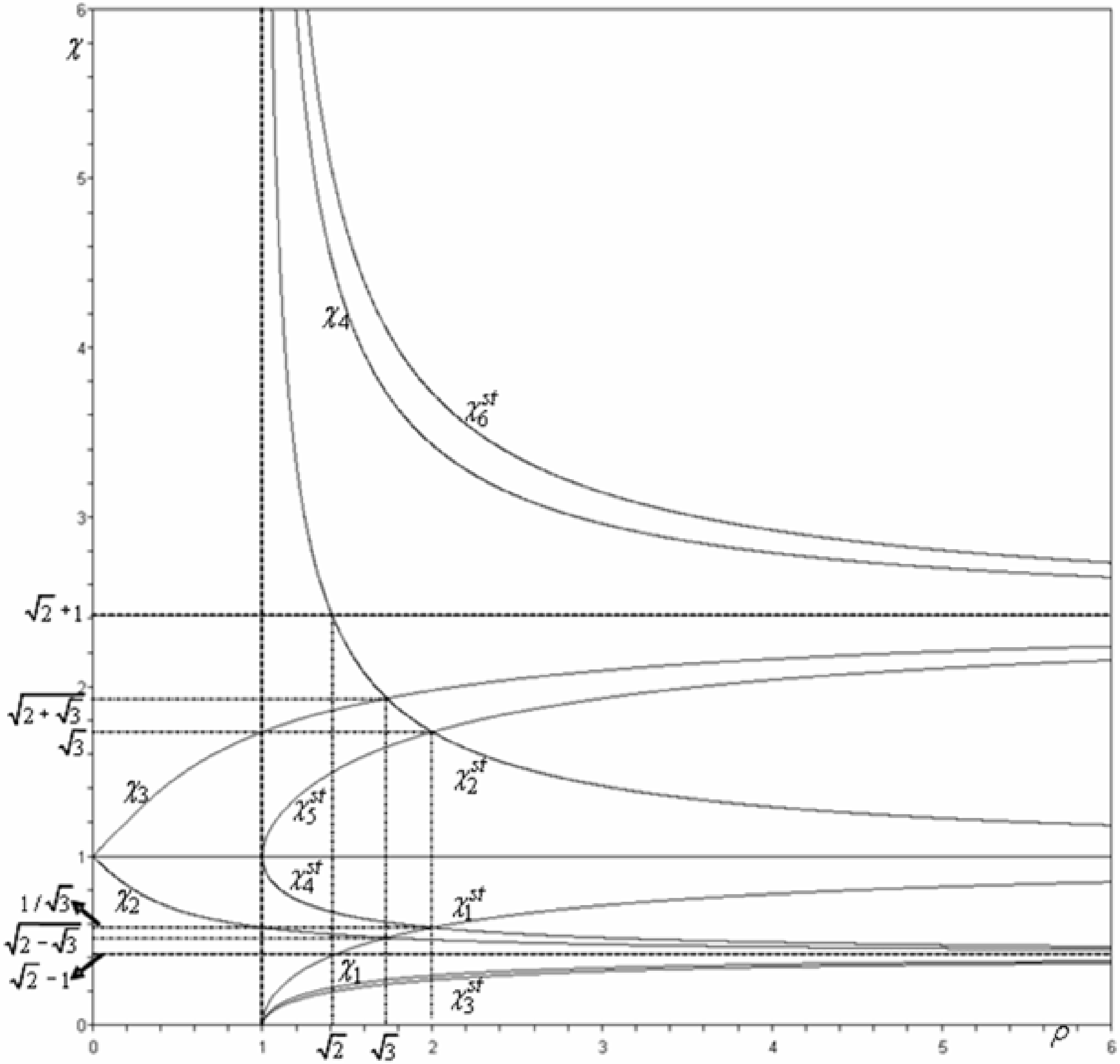
These curves are:

$$\chi = 0, \quad \chi = 1, \quad \chi \to \infty, \quad \chi_1^{st} = \sqrt{\frac{\rho - 1}{\rho + 1}}, \quad \chi_2^{st} = \sqrt{\frac{\rho + 1}{\rho - 1}},$$

$$\chi_3^{st} = \sqrt{\frac{3\rho + 1 - \sqrt{8\rho(\rho + 1)}}{\rho - 1}}, \quad \chi_4^{st} = \sqrt{\frac{3\rho - 1 - \sqrt{8\rho(\rho - 1)}}{\rho + 1}},$$

$$\chi_5^{st} = \sqrt{\frac{3\rho - 1 + \sqrt{8\rho(\rho - 1)}}{\rho + 1}}, \quad \chi_6^{st} = \sqrt{\frac{3\rho + 1 + \sqrt{8\rho(\rho + 1)}}{\rho - 1}}.$$

Bounds on optimal and anti-optimal solutions



Bounds on optimal and anti-optimal solutions

Comparison with $q_{xx}(\theta)$:

$$q_{xx}(\theta)=\frac{Q_{xx}(\theta)}{\sqrt{R_0^2+R_1^2}}=\tau+\frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k\rho\xi_0+4\xi_1\right]\quad \xi_0=\cos 4\theta,\ \xi_1=\cos 2\theta.$$

So, actually φ is similar to $q_{xx}(\theta)$, the isotropic part is the same (τ), and only functions $c_0(\chi)$ and $c_1(\chi)$ introduce the influence of geometry and mode in φ. In particular, the maximum and minimum can be only

$$q_{11}=q_{xx}(\xi_1=1)=\tau+\frac{(-1)^k\rho+4}{\sqrt{1+\rho^2}},$$

$$q_{22}=q_{xx}(\xi_1=-1)=\tau+\frac{(-1)^k\rho-4}{\sqrt{1+\rho^2}},$$

$$q_{\theta\theta}=q_{xx}\left(\xi_1=-(-1)^k/\rho\right)=\tau-(-1)^k\frac{\rho^2+2}{\rho\sqrt{1+\rho^2}},\quad \text{only for } \rho>1.$$

also similar to φ_{11}, φ_{22} and $\varphi_{\delta\delta}$.

Bounds on optimal and anti-optimal solutions

In particular, we have that the extreme values of φ and $q_{xx}(\theta)$ are equal on some of the preceding curves:

$$q_{11} \quad \text{on} \quad \chi = 0 \quad \text{and} \quad \chi \to \infty;$$

$$q_{22} \quad \text{on} \quad \chi = 0 \quad \text{and} \quad \chi \to \infty;$$

$$q_{\theta\theta} \quad \text{on} \quad \chi = 0, \chi \to \infty, \chi = \chi_1^{st} \text{ and } \chi = \chi_2^{st};$$

To complete the comparison with φ, let us introduce the following intermediate values of $q_{xx}(\theta)$:

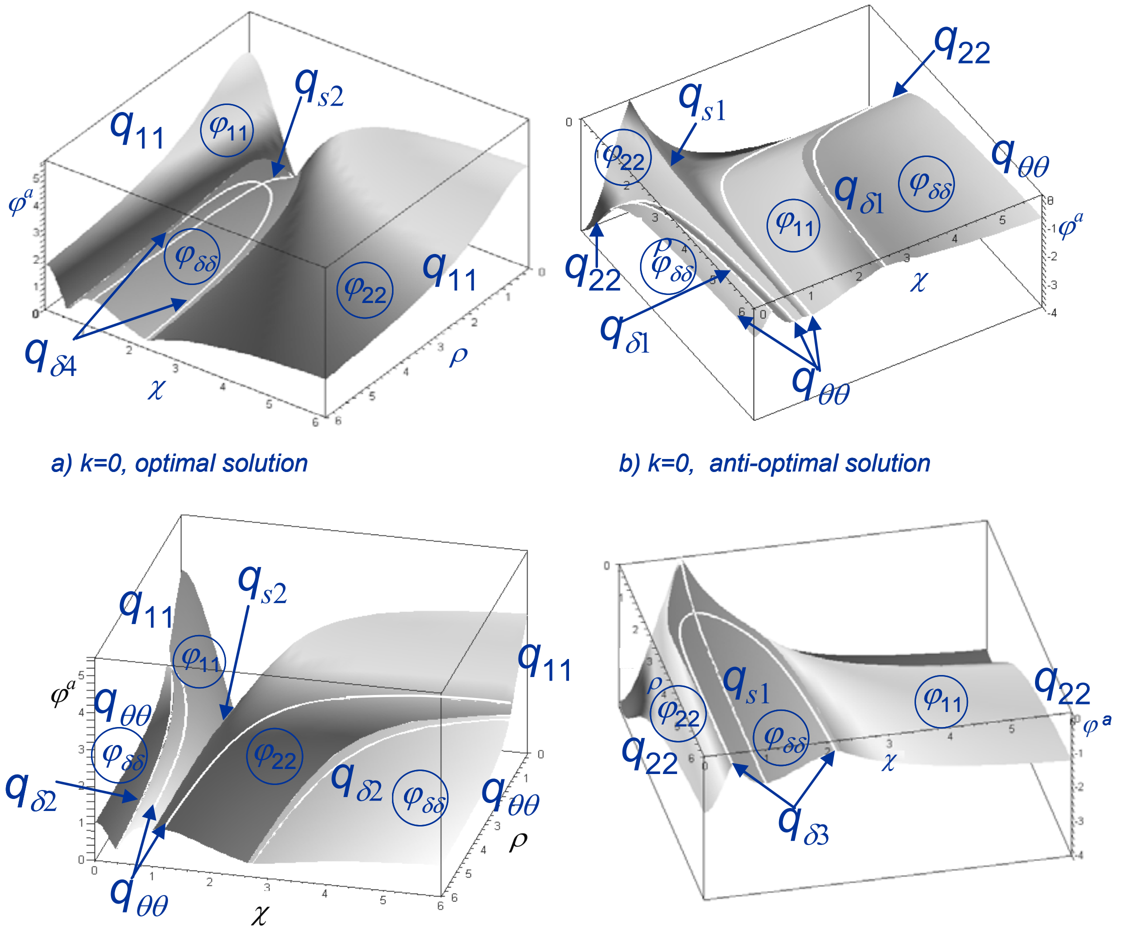
$$q_{s1} = \tau - \frac{\rho}{\sqrt{1+\rho^2}}, \quad q_{s2} = \tau + \frac{\rho}{\sqrt{1+\rho^2}}, \quad \text{on } \chi = 1;$$

$$q_{\delta 1} = \tau - \frac{2\rho+1}{\rho\sqrt{1+\rho^2}}, \quad q_{\delta 2} = \tau + \frac{2\rho+1}{\rho\sqrt{1+\rho^2}}, \quad \text{on } \chi_3^{st} \text{ and } \chi_6^{st};$$

$$q_{\delta 3} = \tau + \frac{2\rho-1}{\rho\sqrt{1+\rho^2}}, \quad q_{\delta 4} = \tau - \frac{2\rho-1}{\rho\sqrt{1+\rho^2}}, \quad \text{on } \chi_4^{st} \text{ and } \chi_5^{st}.$$

Bounds on optimal and anti-optimal solutions

The diagrams of φ^a, the anisotropic part of the optimal solution:



a) k=0, optimal solution

b) k=0, anti-optimal solution

c) k=1, optimal solution

d) k=1, anti-optimal solution

Bounds on optimal and anti-optimal solutions

Remark: q_{11}, q_{22} and $q_{\theta\theta}$, bound the optimal and anti-optimal values of $\varphi(\xi_0, \xi_1)$, φ_{11}, φ_{22} and $\varphi_{\delta\delta}$.

So, global maxima and minima can be taken only for χ =0, $\chi \to \infty$, $\chi = \chi_1^{st}$ and $\chi = \chi_2^{st}$, while local maxima and minima only for χ=1 and on χ_3^{st} to χ_6^{st}.

In addition, χ=1 and χ_3^{st} to χ_6^{st} can be absolute maxima of the anti-optimal solution and absolute minima of the optimal solution.

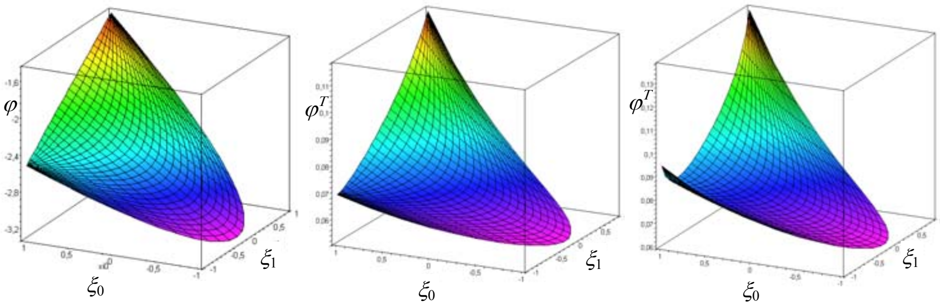
Finally, the optimal and anti-optimal values of the objective function, can be interpreted as a sort of normal stiffness, that takes its highest or lowest possible value in some special cases.

The case of a non-sinusoidal load

In this case the problem of stiffness maximization is

$$\text{minimize} \quad \varphi^T(\xi_0,\xi_1)=\sum_{m,n=1}^{\infty}\frac{p_{mn}^{*2}}{m^4(1+\chi^2)^2\varphi(\xi_0,\xi_1)},$$

$$\text{subjectedto} \quad -1\leq\xi_1\leq 1,\quad 2\xi_1^2-1\leq\xi_0\leq 1.$$



Now, the objective function is no more linear and the isotropic part, τ, influences the location of the solution.

Nevertheless, being

$$\nabla\varphi^T(\xi_0,\xi_1)=\sum_{m,n=1}^{\infty}\frac{p_{mn}^{*2}}{m^4(1+\chi^2)^2\varphi^2(\xi_0,\xi_1)}\left(\frac{(-1)^k\rho\, c_0(\chi)}{\sqrt{1+\rho^2}};\frac{4c_1(\chi)}{\sqrt{1+\rho^2}}\right)\neq(0,0),$$

the solution is again on the boundary (except some pathological situations, see below)

The case of a non-sinusoidal load

A fundamental question is : if we consider the case m=n=1, for which we can find the solution as seen, how much does the optimal solution change when the whole series is considered? To evaluate this, we introduce the ratio

$$\sigma = \frac{p^{*2}_{mn}}{p^{*2}_{11}} \frac{(1+\eta^2)^2}{m^4\left(1+\frac{\eta^2}{\gamma^2}\right)^2} \frac{\varphi_{11}}{\varphi_{mn}},$$

It can be seen that for $m\rightarrow\infty$, $\sigma\rightarrow 0\ \forall\gamma$; in addition, for γ=1, *i.e*. if m=n, then

$$\sigma = \frac{p^{*2}_{mn}}{m^4 p^{*2}_{11}},$$

For instance, for a uniform load $\sigma=1/m^8$, *viz*. the term m=n=3 is $1/6561\approx 1.5\times 10^{-4}$ the first term, while for a concentrated load it is $1/324\approx 3.1\times 10^{-3}$.

The case of a non-sinusoidal load

Generally, though not quite identical, optimal solutions for a generic load do not differ substantially form those found for a sinusoidal load; this means also that optimal solutions for buckling and natural frequencies for $\chi=\eta$ are similar to those of bending stiffness for a generic load.

Optimal orientation angle δ	Fir wood	Boron-epoxy B(4)-55054	Carbon-epoxy T300-5208	Glass-epoxy balanced fabric	Braided carbon-epoxy BR45a
		η=1.2			η=6.0
δ_{sin}	52.16°	50.40°	51.05°	45°	70.64°
δ_{unif}	51.86°	50.22°	50.82°	45°	75.68°
δ_{conc}	49.52°	48.65°	48.92°	45°	90.00°
$\delta_{unif}-\delta_{sin}$	–0.30°	–0.18°	–0.23°	0°	5.04°
$\delta_{conc}-\delta_{sin}$	–2.66°	–1.75°	–2.12°	0°	19.36°

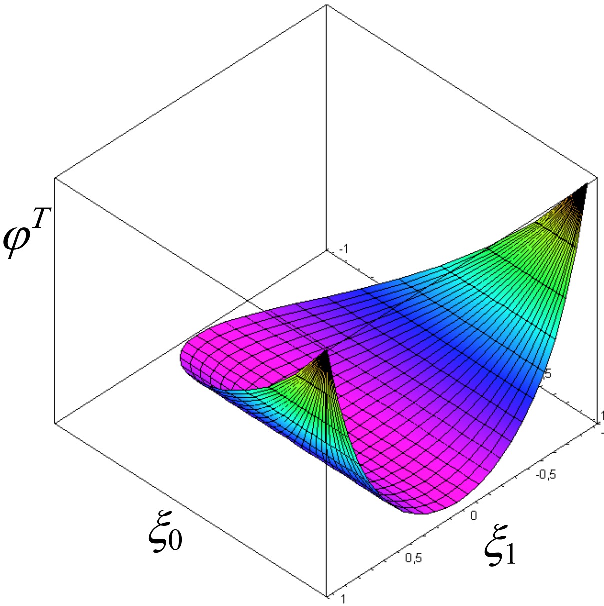
(m_{max}=n_{max}=50)

The case of a non-sinusoidal load: pathological solutions

The previous pathological solutions do not exist, as χ changes with *m* and *n.*

Nevertheless, if ρ =0, functions φ and φ^T do not depend upon ξ_0. In such a case, $\nabla\varphi^T$ can be null along a line parallel to the ξ_0 axis.

For η=1,possible solutions are a balanced cross-ply laminate as well as an angle-ply with δ=45°, but also any other laminate of the type (ξ_1=0, $-1\leq \xi_0 \leq 1$), for instance isotropic laminates in bending(ξ_1=0, ξ_0=0).

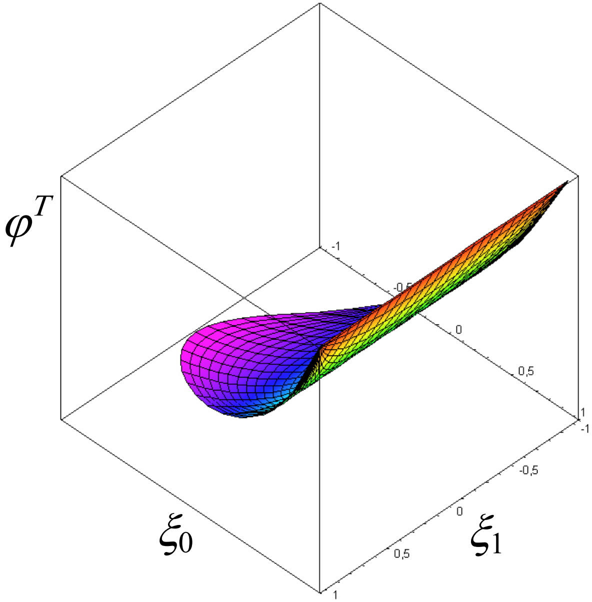


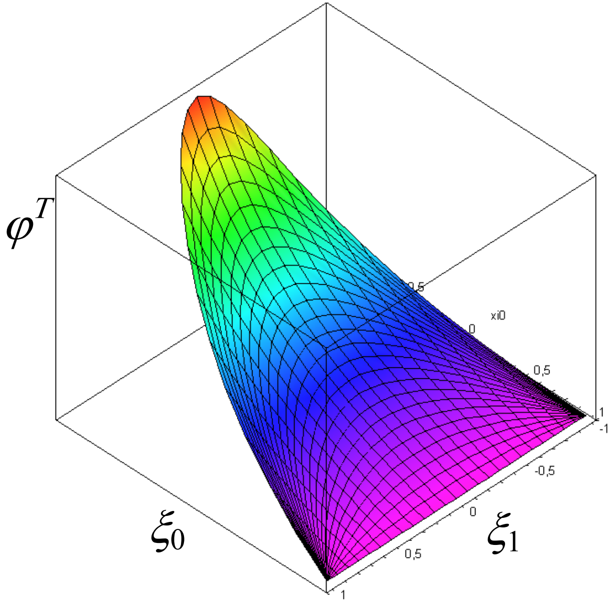
The case of a non-sinusoidal load: pathological solutions

The other case is $\rho \to \infty$; φ and φ^T do not depend upon ξ_1.

$\nabla \varphi^T$=0 for ξ_0=±1, that is for an angle ply with δ=45°, if *k*=0, or for a cross-ply laminate, if *k*=1.

As for a sinusoidal load, all the cross-ply laminates are possible solutions, if *k*=1.





The optimal critical load of buckling

The problem is to determine the mode (m, n) that leads to the lowest buckling load.

To this purpose, we consider the ratio

$$\varepsilon = \frac{\lambda_a}{\lambda_b} = \left(\frac{m_a}{m_b}\right)^2 \left(\frac{\gamma_a^2 + \eta^2}{\gamma_b^2 + \eta^2}\right)^2 \frac{\gamma_b^2 + \nu\eta^2}{\gamma_a^2 + \nu\eta^2} \frac{\varphi_a}{\varphi_b},$$

where λ_a is the buckling load for $m=m_a$, $n=n_a$, and λ_b for $m=m_b$, $n=n_b$; φ_a is φ calculated for η/γ_a and φ_b for η/γ_b.

If $\gamma_a=\gamma_b$, then $\varepsilon = \left(\frac{m_a}{m_b}\right)^2$ $\rightarrow$ we can consider $m_b=n_b=1$ and analyse what happens for a given m_a and for $\gamma_a\neq 1$.

The optimal critical load of buckling

The two buckling loads $m_b=n_b=1$ and (m_a, n_a) can be found and ε computed; so, for a given plate and mode (m_a), we can trace the surface representing $\varepsilon(\gamma_a, \nu)$ and look when it is lower or greater than 1. The curves separating the domains can be put in explicit form; they are:

$b_1 \rightarrow$ solution of $\varepsilon \rightarrow \infty$: $\nu = -\dfrac{\gamma_a^2}{\eta^2}$;

$b_2 \rightarrow$ solution of $\varepsilon=1$: $\nu = -\dfrac{1}{\eta^2}\dfrac{m_a^2 - A\gamma_a^2}{m_a^2 - A}, \quad A = \dfrac{\varphi}{\varphi_a}\dfrac{(1+\eta^2)^2}{(\gamma^2+\eta^2)^2}$;

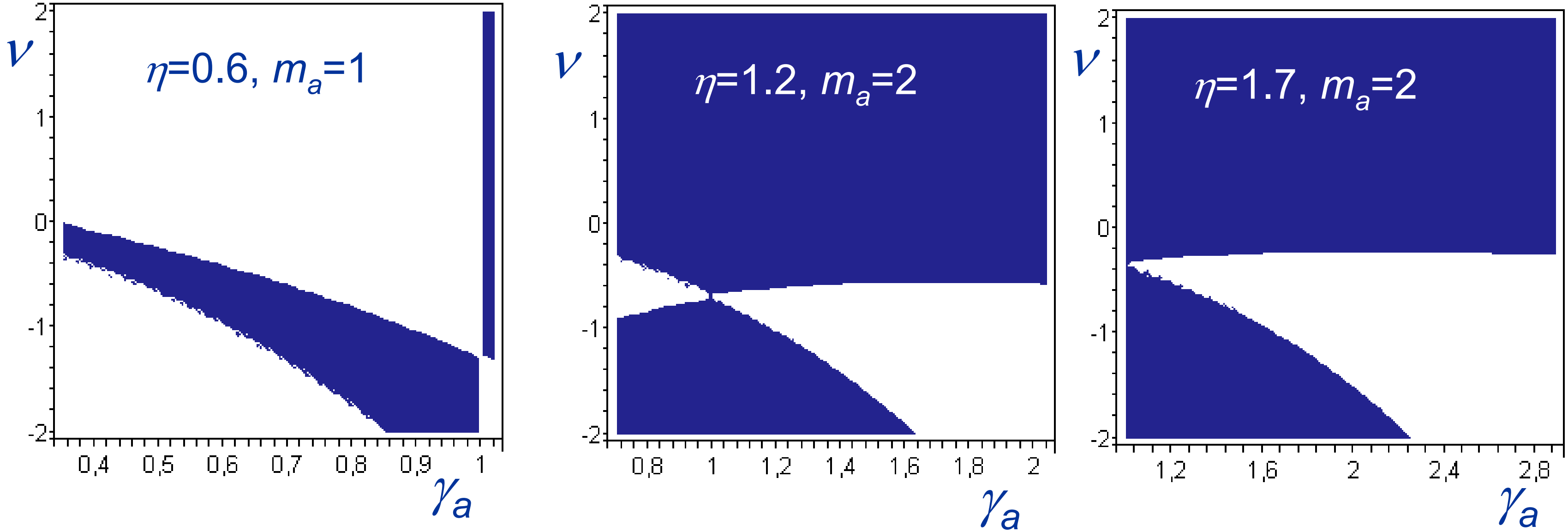
$b_3 \rightarrow$ trivial solution $\gamma_a=1$ if $m_a=1$.

The optimal critical load of buckling

The areas where ε>1 are those where the buckling load of the case *a* is greater than the one of the case *b*.

In this way all the significant cases can be easily verified.

Examples: carbon-epoxy, τ=2.62, ρ=0.92, *k*=0 (ε>1 in blue).



The optimal fundamental frequency

Like for the critical load, the problem is to determine the mode (m, n) that leads to the lowest natural frequency.

To this purpose, we consider the ratio

$$\varpi = \left(\frac{\omega_a}{\omega_b}\right)^2 = m^{*4}\left(\frac{1+\chi^{*2}}{1+\chi^2}\right)^2 \frac{\varphi(\chi^*)}{\varphi(\chi)},$$

where ω_a is the natural frequency for $m=m_a$, $n=n_a$, and ω_b for $m=m_b$, $n=n_b$; $\varphi(\chi)$ is φ calculated for $\chi=\eta/\gamma_b$ and $\varphi(\chi^*)$ for $\chi^*=\eta/\gamma_a=\chi/\gamma^*$, where $\gamma^*=\gamma_a/\gamma_b$ and $m^*=m_a/m_b$.

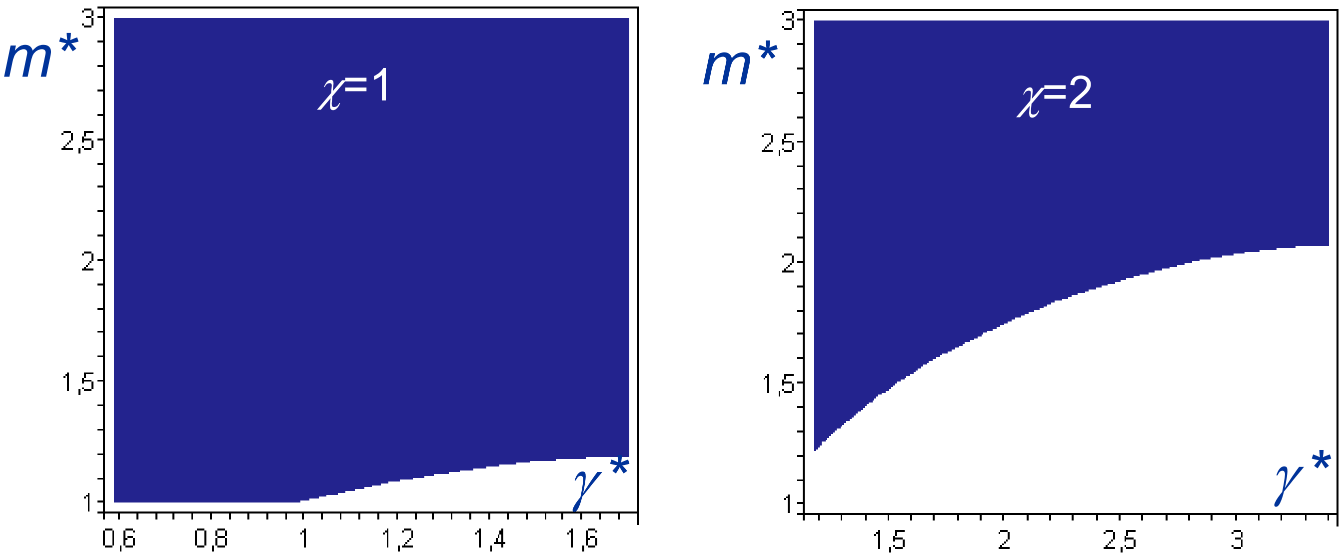
We can trace the curve $\varpi(m^*,\gamma^*)=1$ and see where $\varpi>1$.

The optimal fundamental frequency

The equation of the curve is simply

$$m^* = \left[\left(\frac{1+\chi^{*2}}{1+\chi^2} \right)^2 \frac{\varphi(\chi^*)}{\varphi(\chi)} \right]^{-4}.$$

Examples: carbon-epoxy, τ=2.62, ρ=0.92, k=0 (ϖ>1 in blue).



Some examples of exact optimal solutions

A simple strategy for obtaining exact specially orthotropic laminates in bending: to choose a *quasi-homogeneous solution* ($\mathbf{A}/h=12\mathbf{D}/h^3$, $\mathbf{B}=\mathbf{O}$) of the *quasi-trivial* set.

This ensures that for the angle-ply laminate will be not only $\mathbf{B}=\mathbf{O}$, but also $D_{xs}=D_{ys}=0$ as they are equal to A_{xs} and A_{ys} which are automatically null for angle-ply laminates.

So, for this class of laminates, the Navier's solutions are exact.

Example 1: carbon-epoxy laminate, $\tau=2.62$, $\rho=0.92$, $k=0$, $\eta=1.2$, $\gamma=1$, $\nu=1$.

As $1<\chi=1.2<\chi_3=1.70$ and $k=0$, the optimal value of the objective function is $\varphi_{\max}=\varphi_{\delta\delta}=3.31$.

Some examples of exact optimal solutions

The solution angle is δ=51°.

The gain with respect to the intuitive solution φ_{22}=2.57 is ζ_{max}=1.29.

The optimal solution for bending stiffness in case of uniform load is δ=50.8° and for concentrated load 48.9°.

Example 2: braided carbon-epoxy BR45a laminate, τ=6.01, ρ=2.03, k=1, η=10, γ=1, ν=1.

As χ=10>χ_4=3.40, ρ>1 and k=1, the optimal value of the objective function is $\varphi_{\max}=\varphi_{\delta\delta}$=7.29.

Some examples of exact optimal solutions

The solution angle is δ=60.8°.

The gain with respect to the intuitive solution φ_{22}=6.06 is ζ_{max}=1.2.

The optimal solution for bending stiffness in case of uniform load is δ=61.5° and for concentrated load 65.9°.

Possible exact (unsymmetrical) solutions:

8 plies: $[\delta, -\delta, -\delta, \delta, -\delta, \delta, \delta, -\delta]$

12 plies: $[\delta, -\delta, \delta, -\delta_3, \delta_3, -\delta, \delta, -\delta]$

16 plies: $[\delta, -\delta, \delta, -\delta_2, \delta, -\delta_2, \delta_4, -\delta_3, \delta]$ etc.

Conclusion

The design analysis made with dimensionless invariant parameters helps in some way the understanding of the flexural problems in presence of anisotropy: it puts in evidence some pathological situations and characterizes the localization of the different types of optimal solutions as well as their effectiveness.

This study is merely qualitative, but can help in similar studies with other geometries and conditions.

Thank you very much for your attention.