Parametric Resonance Problems

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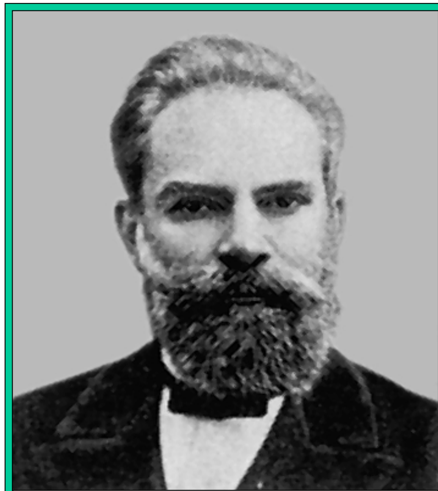
Key questions

- *Instability regions for Hill's equation with damping*
- *Inverted pendulum: influence of damping and arbitrary periodic excitation function*
- *A swing – one of the simplest parametric resonance problems*
- *Instability regions for a system with varying moment of inertia*
- *General case of a system with finite degrees of freedom*

Method. Stability analysis is based on derivatives of the Floquet matrix with respect to problem parameters.

Periodical Systems



Alexander M. Liapunov
1857 – 1918

Thesis (1892)
«General problem on stability of motion»

Chapter 3
Study of periodic movements

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}$$

Introduction of parameters

$$\mathbf{G}(t,\mathbf{p}) = \mathbf{G}(t+T,\mathbf{p})$$

p – vector of parameters

Stability of periodic systems

General stability theory by Floquet (1883)

$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \quad \mathbf{G}(t) = \mathbf{G}(t+T)$

matriciant

$\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}$

Monodromy matrix

$\mathbf{F} = \mathbf{X}(T)$

multipliers ρ

$\mathbf{Fu} = \rho \mathbf{u}$

Asymptotic stability $|\rho| < 1$ | Instability $|\rho| > 1$

New results

$$\frac{\partial \mathbf{F}}{\partial p_j} = \mathbf{F} \int_0^T \mathbf{X}^{-1} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X} dt$$

$$\frac{\partial \rho}{\partial p_j} = \rho \mathbf{v}^T \int_0^T \mathbf{X}^{-1} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X} \mathbf{u} dt$$

p_j – *a system parameter*

Theory of bifurcations of multipliers

Hill's equation with damping

$$\ddot{y} + \gamma \dot{y} + (\omega^2 + \delta \varphi(t)) y = 0$$

$$\varphi(t) = \varphi(t + 2\pi)$$

Three parameters $\mathbf{p} = (\delta, \gamma, \omega)$: small amplitude and damping $\delta, \gamma << 1$, arbitrary frequency ω

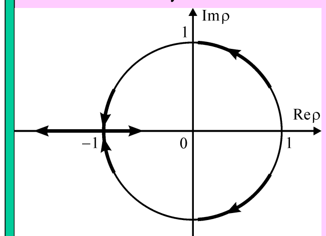
First order form: $\mathbf{x} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \quad \mathbf{G}(t, \mathbf{p}) = \begin{bmatrix} 0 & 1 \\ -\omega^2 - \delta \varphi(t) & -\gamma \end{bmatrix}$

The matriciant for $\delta = \gamma = 0$

$$\mathbf{X_0}(t) = \begin{bmatrix} \cos \omega t & \omega^{-1} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

Monodromy matrix and multipliers for $\delta = \gamma = 0$

$$\mathbf{F_0} = \mathbf{X_0}(2\pi) = \begin{bmatrix} \cos 2\pi\omega & \omega^{-1} \sin 2\pi\omega \\ -\omega \sin 2\pi\omega & \cos 2\pi\omega \end{bmatrix}$$



$$\rho_{1,2} = \cos 2\pi\omega \pm i \sin 2\pi\omega$$

Simple complex conjugate multipliers

$\omega \neq k/2, \; k = 1, 2, \ldots \qquad |\rho_{1,2}| = 1$

Liouville formula

$$\rho_1 \rho_2 = \exp\left(\int_0^T Tr(\mathbf{G}) dt \right) = \exp(-2\pi\gamma) \leq 1$$

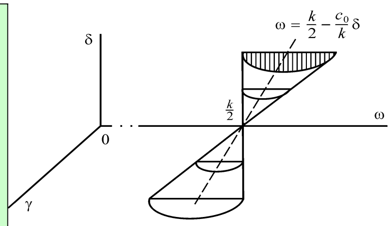
Instability can take place near the points

$\mathbf{p_0} : \; \delta = 0, \; \gamma = 0, \; \omega = k/2, \; k = 1, 2, \ldots$

Instability regions for Hill's equation with damping

Taylor's series near $\mathbf{p_0}$:

$$\mathbf{F(p)} = \cos \pi k \times \begin{pmatrix} 1 + \frac{\pi \delta b_k}{k} - \pi\gamma & 4\pi\left(\Delta\omega + \frac{(2c_0 - a_k)\delta}{2k^2}\right) \\ -\pi k \left(\Delta\omega + \frac{(2c_0 + a_k)\delta}{2k}\right) & 1 - \frac{\pi \delta b_k}{k} - \pi\gamma \end{pmatrix}$$



Fourier coefficients

$$a_k = \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \cos kt dt, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \sin kt dt,$$

$$r_k = \sqrt{a_k^2 + b_k^2}, \quad c_0 = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) dt$$

Analytical description of regions of parametric resonance:

$$4\left(\Delta\omega + \frac{c_0}{k}\delta\right)^2 + \gamma^2 < \frac{r_k^2}{k^2}\delta^2$$

Stability and instability of periodic solutions of nonlinear systems

Duffing's equation: $\ddot{u} + 2\mu\dot{u} + \omega_0^2 u + \alpha u^3 = f\cos\Omega t$

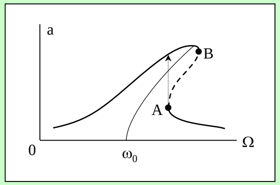
Periodic solution: $u_0(t) = a\cos(\Omega t - \gamma) + \dfrac{\alpha a^3}{32\omega_0^2}\cos(3\Omega t - 3\gamma)$

Perturbed solution: $u(t) = u_0(t) + v(t)$

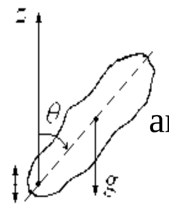
Damped Hill's equation $\ddot{v} + 2\mu\dot{v} + [\omega_0^2 + 3\alpha a^2\cos^2(\Omega t - \gamma)]v = 0$

Instability condition:

$$\left(\Omega - \omega_0 - \frac{3\alpha a^2}{8\omega_0}\right)\left(\Omega - \omega_0 - \frac{9\alpha a^2}{8\omega_0}\right) + \mu^2 < 0$$



Stability of inverted pendulum with excitation of the pivot



What is new: damping and arbitrary periodic function

$z = a\phi(\Omega t)$

$I\ddot{\theta} + c\dot{\theta} - mr(g + \ddot{z})\sin\theta = 0$

Stephenson (1908)
Kapitza (1951)

Non-dimensional variables

$\beta = \dfrac{c}{I\Omega} \quad \varepsilon = \dfrac{a\Omega_0^2}{g} \quad \omega = \dfrac{\Omega_0}{\Omega}$

- small parameters

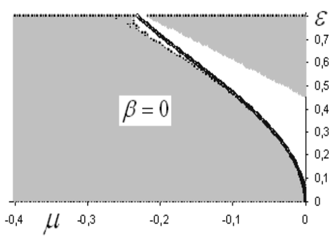
Hill's equation with damping

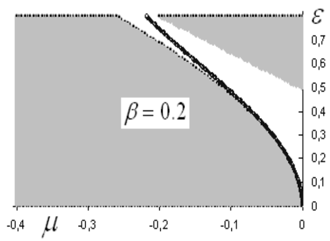
$\ddot{\theta} + \beta\dot{\theta} - [\omega^2 - \varepsilon\varphi(\tau)]\theta = 0$

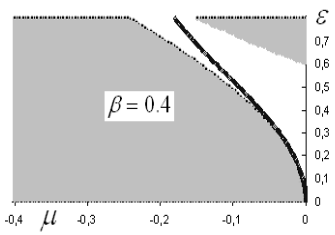
$\varphi(\tau) = \cos\tau$ Stabilization condition $-\omega^2 > -\dfrac{\varepsilon^2}{2} + \dfrac{\varepsilon^2\beta^2}{2} + \dfrac{7\varepsilon^4}{32}$

Instability regions for Mathieu-Hill equation with damping

$$\ddot{\theta} + \beta\dot{\theta} + [\mu + \varepsilon\varphi(\tau)]\theta = 0, \quad \varphi = \cos\tau$$

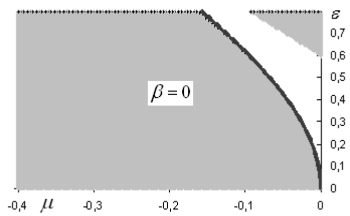


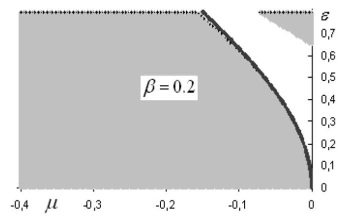


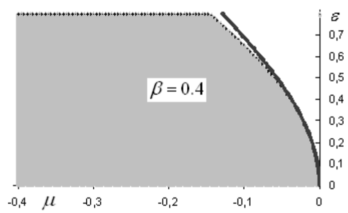


Comparison between analytical and numerical results

Instability regions for the case $\varphi = \cos^3\tau$







Destabilization effect of small damping

Stabilization frequency for the pendulum

General formula for symmetric functions $\varphi(\tau+\pi)=-\varphi(\tau)$

$$\frac{\Omega}{\Omega_0} > \frac{1}{\varepsilon\sqrt{-F}} - \frac{L\varepsilon}{2F\sqrt{-F}} + \frac{K\varepsilon\beta_0^2}{2\sqrt{-F}} + \left(\frac{3L^2}{8F^2\sqrt{-F}} - \frac{H}{2F\sqrt{-F}}\right)\varepsilon^3$$

$$F = \left(\frac{1}{2\pi}\int_0^{2\pi} t\varphi(t)dt\right)^2 - \frac{1}{\pi}\int_0^{2\pi}\varphi(t)\int_0^t \tau\varphi(\tau)\,d\tau\,dt < 0$$

Stabilization frequency for the pendulum

For symmetric function $\varphi(\tau)=\cos\tau$

$$\frac{\Omega}{\Omega_0} > \sqrt{2}\left[\frac{1}{\varepsilon} + \frac{7\varepsilon}{32} + \frac{\varepsilon\beta_0^2}{4} - \frac{2389\,\varepsilon^3}{18432}\right]$$

For non-symmetric function $\varphi(\tau)=\frac{1}{4}\left(\frac{\tau}{\pi}\right)^3-\frac{1}{2}$

$$\frac{\Omega}{\Omega_0} > \frac{2.19}{\varepsilon} - 0.202 + 0.162\varepsilon + 0.045\varepsilon^2 + 0.214\varepsilon\beta_0^2 - 0.028\varepsilon^3$$

A swing – simplest model for parametric resonance

Nonlinear system - a pendulum of variable length:

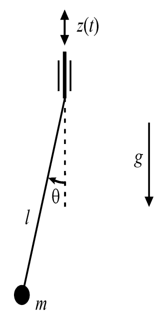
$$(ml^2\dot{\theta})^{\cdot} + \gamma l^2\dot{\theta} + mgl\sin\theta = 0$$

$$l(t) = l_0 + a\varphi(\Omega t)$$

Resonant frequencies:

$$\Omega_k = \frac{2}{k}\sqrt{\frac{g}{l_0}},$$

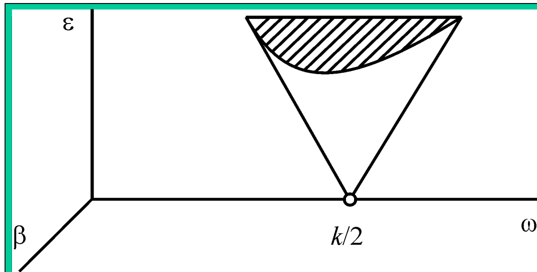
$$k = 1,2,\ldots$$



Non-dimensional parameters

$$\varepsilon = \frac{a}{l_0},\ \beta = \frac{\gamma}{m\sqrt{g/l_0}},\ \omega = \frac{\sqrt{g/l_0}}{\Omega}$$

Instability regions for the swing

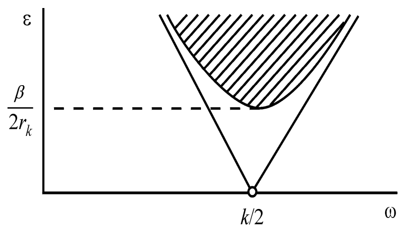


A half-cone in physical parameter space

$$(\beta/2)^2 + (2\omega/k - 1)^2 < r_k^2\varepsilon^2,\ \beta \ge 0$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\cos k\tau d\tau,\quad b_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\sin k\tau d\tau,\quad r_k = \frac{3}{4}\sqrt{a_k^2 + b_k^2}$$

Projections of instability regions

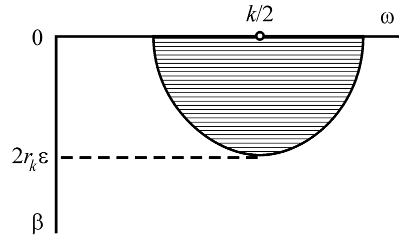


$$\beta = const \geq 0 - hyperbola$$

Minimal instability amplitude

$$\varepsilon_{\min} = \beta/(2r_k) \to \infty$$

$$when \;\; k \to \infty$$

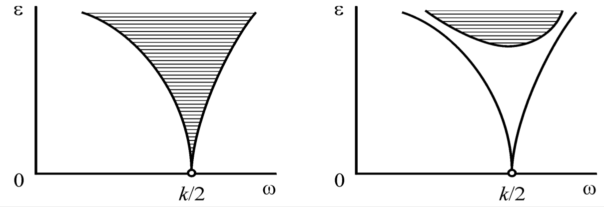


$$\varepsilon = const - ellipse$$

No resonance

at the damping $\beta > 2r_k\varepsilon$

Degenerate case $r_k = 0$



$$\varepsilon_{\min} = \sqrt{\frac{\beta}{R_k}}$$

Resonance regions:

$$\beta^2 + 4\left(\frac{2\omega}{k} - 1 + \frac{\eta_k \varepsilon^2}{2}\right)^2 < R_k^2 \varepsilon^4, \;\; \beta \geq 0\,, \;\; R_k = \sqrt{\xi_k^2 + \zeta_k^2}$$

$$\xi_k = -\frac{3}{2\pi}\int_0^{2\pi} \varphi^2(t)\sin kt\,dt + \frac{3k}{4\pi}\int_0^{2\pi} \varphi(t)\int_0^t \varphi(\tau)\cos k\tau\,d\tau\,dt$$

$$\zeta_k = \frac{3}{2\pi}\int_0^{2\pi} \varphi^2(t)\cos kt\,dt + \frac{3k}{4\pi}\int_0^{2\pi} \varphi(t)\int_0^t \varphi(\tau)\sin k\tau\,d\tau\,dt$$

Examples

Excitation function: $\varphi(\tau) = \begin{cases} 1, & 0 \leq \tau \leq \pi \\ -1, & \pi < \tau \leq 2\pi \end{cases}$

$$a_{2k-1} = 0, \qquad b_{2k-1} = \frac{4}{\pi(2k-1)}, \qquad r_{2k-1} = \frac{3}{\pi(2k-1)}.$$

Resonance regions: Magnus (1976)

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{\Omega_{2k-1}} - 1\right)^2 < \frac{9a^2}{\pi^2 l_0^2 (2k-1)^2}, \qquad k = 1,2,\ldots$$

Degenerate case

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{\Omega_{2k}} - 1 + \frac{3}{8}\left(\frac{a}{l_0}\right)^2\right)^2 < \frac{9}{16}\left(\frac{a}{l_0}\right)^4, \qquad k = 1,2,\ldots$$

Examples

Excitation function: $\varphi(\tau) = \cos\tau - \sin 2\tau$

$$a_1 = 1, \quad b_2 = -1, \quad a_2 = b_1 = 0, \qquad r_1 = r_2 = 3/4$$

$$a_k = b_k = r_k = 0, \qquad k = 3,4\ldots$$

First resonance region

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{2\sqrt{g/l_0}} - 1\right)^2 < \frac{9a^2}{16 l_0^2}$$

Second resonance region

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{\sqrt{g/l_0}} - 1\right)^2 < \frac{9a^2}{16 l_0^2}$$

Instability of a system with periodically varying moment of inertia

Moving masses

$$r = r_0 + a\,\varphi(\Omega t)$$

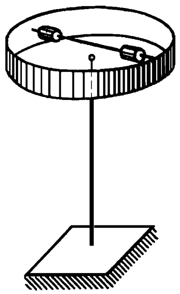
Moment of inertia

$$J(t) = J_0 + 2m[r_0 + a\varphi(\Omega t)]^2$$

Twisting oscillations

$$(J(t)\dot{\theta})^{\cdot} + \gamma\dot{\theta} + c\theta = 0$$

Small damping γ and amplitude a



Vertical elastic shaft with a hard disk and two moving masses

First order equations with four parameters

Non-dimensional variables and parameters

$$\tau = \Omega t \qquad \varepsilon = \frac{a}{r_0}$$

$$\beta = \frac{\gamma}{\sqrt{\tilde{J}_0 c}} \qquad \omega = \frac{1}{\Omega}\sqrt{\frac{c}{\tilde{J}_0}}$$

$$\varsigma = \frac{2mr_0^2}{\tilde{J}_0}, \qquad \tilde{J}_0 = J_0 + 2mr_0^2$$

$$x_1 = \theta \qquad x_2 = \frac{\tilde{J}(t)\dot{\theta}}{\Omega}$$

$$\tilde{J}(t) = \frac{J(t)}{\tilde{J}_0}$$

$$\frac{dx_1}{d\tau} = \frac{1}{\tilde{J}(\tau)}x_2$$

$$\frac{dx_2}{d\tau} = -\omega^2 x_1 - \frac{\beta\omega}{\tilde{J}(\tau)}x_2$$

$$\tilde{J}(\tau) = 1 + 2\varepsilon\varsigma\varphi(\tau) + \varepsilon^2\varsigma\varphi^2(\tau)$$

Four parameters $\varsigma, \omega, \varepsilon, \beta$

$$0 \le \varepsilon << 1 \qquad 0 \le \beta << 1 \qquad 0 < \varsigma < 1$$

Instability regions

Parametric resonance:

$$4\,(2\omega/k - 1)^2 + \beta^2 < r_k^2\varepsilon^2\varsigma^2$$

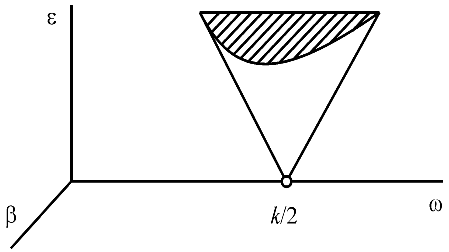
Minimal amplitude

$$\varepsilon_{\min} = \frac{\beta}{r_k\varsigma}$$

With growing k $\quad \varepsilon_{\min} \to \infty$

Impossible to observe resonance at high k !

Half-cone



Increasing instability region with the growing parameter

$$\varsigma = 2mr_0^2/(J_0 + 2mr_0^2)$$

Dimensional quantities

$$\Omega_{cr} = \frac{2\Omega_0}{k}, \quad k = 1, 2, \ldots; \quad \Omega_0 = \sqrt{\frac{c}{J_0 + 2mr_0^2}}$$

General formula for instability regions

$$4\left(\frac{\Omega}{2\Omega_k} - 1\right)^2 + \frac{\gamma^2}{(J_0 + 2mr_0^2)\,c} < \frac{a^2 r_k^2 \varsigma^2}{r_0^2}$$

$\varphi(\tau) = \cos\tau$ $\quad$ First instability region $\quad$ $\varsigma = 1/2$

$$4\left(\frac{\Omega}{2\Omega_0} - 1\right)^2 + \frac{\gamma^2}{(J_0 + 2mr_0^2)\,c} < \frac{a^2}{4r_0^2}$$

Regions of parametric resonance: general case

Vibrational system $(\mathbf{q} \in R^n)$

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma \mathbf{D}\dot{\mathbf{q}} + (\mathbf{P} + \delta \mathbf{B}(\Omega t))\mathbf{q} = 0$$

M > 0, **D** > 0, **P** > 0, **B**(t) = **B**($t + T$)

Parameters:

Ω and δ - frequency and amplitude of parametric excitation

$\gamma > 0$ - damping parameter
(δ and γ - *small parameters*)

Free vibrations ($\delta = \gamma = 0$):

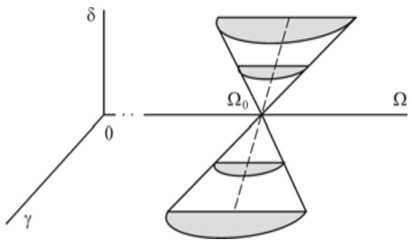
$\omega_1, \dots, \omega_n$ - *eigenfrequencies*
$\mathbf{u}_1, \dots, \mathbf{u}_n$ - *eigenmodes*

1) Main resonances:

$$\Omega \approx 2\omega_i / k$$

2) Sum and difference type of resonances: $\Omega \approx (\omega_i \pm \omega_j)/k$

For $\mathbf{B}(t) = \varphi(t)\mathbf{B}_0$ or $\mathbf{B} = \mathbf{B}^T$ resonance regions are halves of cones



Two important cases:
a) symmetric matrix $\mathbf{B}(\tau) = \mathbf{B}^T(\tau)$
b) $\mathbf{B}(\tau) = \mathbf{B}_0\varphi(\tau)$, $\mathbf{B}_0$ - constant matrix

Half-cones in three-dimensional parameter space $\mathbf{p} = (\delta, \gamma, \Omega)$

$$\eta_j \eta_l \gamma^2 \mp \xi \delta^2 + 4k^2 \frac{\eta_j \eta_l}{(\eta_j + \eta_l)^2}\left(\Delta\Omega + \frac{\sigma_\pm \delta}{k}\right)^2 \le 0$$

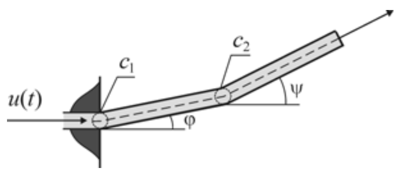
Coefficients

$$\eta_j = \mathbf{u}_j^T \mathbf{D} \mathbf{u}_j, \quad \eta_l = \mathbf{u}_l^T \mathbf{D} \mathbf{u}_l, \quad \xi = \frac{a_k^{jl} a_k^{lj} + b_k^{jl} b_k^{lj}}{4\omega_j \omega_l}$$

$$a_k^{jl} = \frac{1}{\pi}\int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_l \cos k\tau \, d\tau, \quad b_k^{jl} = \frac{1}{\pi}\int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_l \sin k\tau \, d\tau$$

$$\Delta\Omega = \Omega - \frac{\omega_j \pm \omega_l}{k}, \quad \sigma_\pm = -\frac{\omega_j a_0^{jj} \pm \omega_l a_0^{ll}}{4\omega_j \omega_l}$$

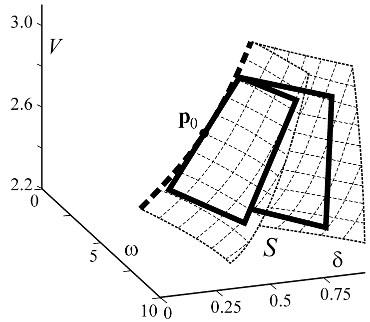
Applications



Stability region and approximation of the singularity (dihedral angle)

Pipe conveying fluid with pulsating speed

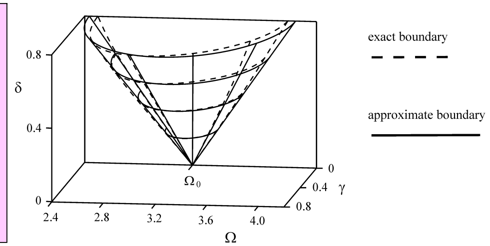
$$u(t) = V(1 + \delta \cos(\Omega t))$$



Bolotin's problem (1956)

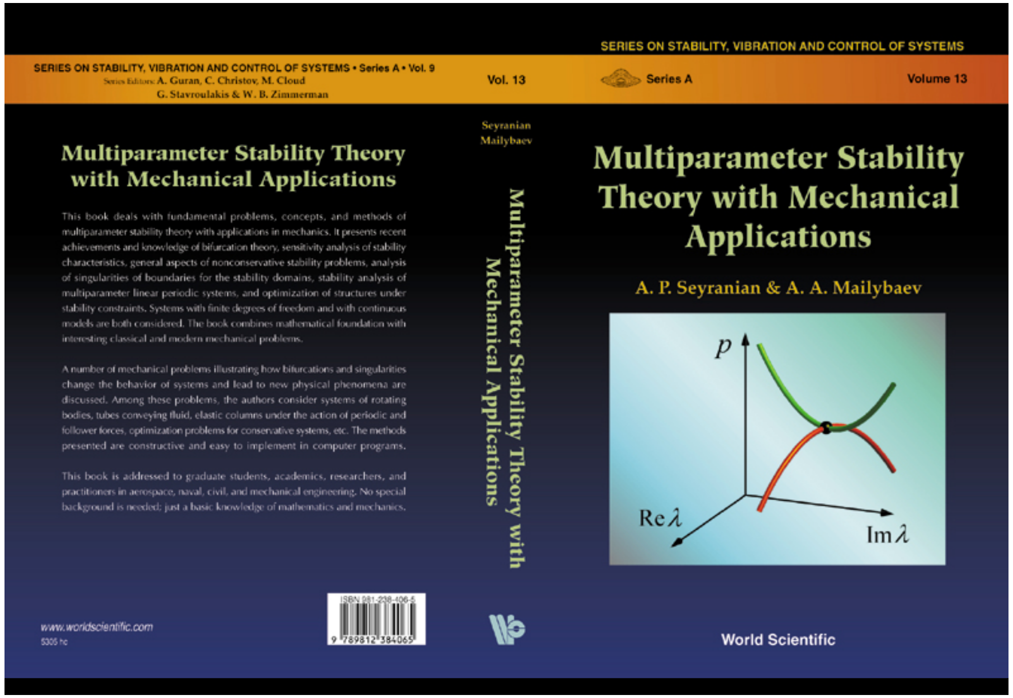
Parametric resonance regions

Stability of a beam loaded by periodic moments



Analytical expression for resonance regions:

$$d_1 d_2 \gamma^2 - \frac{c_{12}(a_k^2 + b_k^2)}{4\omega_{n1}\omega_{n2}}\delta^2 + 4k^2 \frac{d_1 d_2}{(d_1 + d_2)^2}(\Omega - \Omega_0)^2 \le 0$$



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