UNIVERSITÀ DI PISA

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A V V I S O D I S E M I N A R I

Nell'ambito delle iniziative promosse dal ***Dottorato in Ingegneria delle Strutture*** *e dal* ***Dottorato in Ingegneria Aerospaziale****,*

il Prof. **Gianpietro Del Piero**, dell'Università di Ferrara ed
il Prof. **Jean-Jacques Marigo**, dell'Université Paris 6,

terranno i due seminari seguenti:

1) Introduzione alla teoria della frattura di G. Francfort e J.J. Marigo

(Gianpietro Del Piero, ***martedì 13 giugno, ore 17.00****)*

Sommario. *Il seminario ha per oggetto una breve illustrazione della teoria della frattura di un corpo elastico dovuta a G. Francfort ed a J.J. Marigo. Secondo la formulazione data in [Francfort & Marigo, 1998], il problema della frattura è ridotto al problema di minimo di un funzionale energia costituito da due termini: un'energia a densità volumetrica, dipendente in maniera quadratica dal gradiente di deformazione, e un'energia a densità superficiale, proporzionale all'area della superficie di frattura. Seguendo l'evoluzione della deformazione al variare di un parametro di carico, è possibile sia predire la formazione della frattura che seguirne la propagazione.*

Ciò è possibile grazie alla tecnica risolutiva messa a punto in [Ambrosio & Tortorelli, 1990], basata sull'approssimazione del funzionale energia, che è definito sulle funzioni a variazione limitata mediante una famiglia di funzionali più regolari, definiti sugli ordinari spazi di Sobolev. La convergenza dei funzionali approssimanti è assicurata nel senso della Gamma-convergenza. Verranno illustrati alcuni risultati ottenuti in [Bourdin, Francfort & Marigo, 2000] su problemi modello bidimensionali. Si fornirà infine un breve cenno alla recente estensione del modello al problema della fatica [Jaubert & Marigo, in stampa].

2) A variational approach to fracture: the evolution of cracks seen as an energy minimization problem

(Jean-Jacques Marigo, ***mercoledì 14 giugno, ore 11.00****)*

Abstract. *Griffith proposed in his famous paper to base the propagation of cracks on a principle of least energy. Many researchers and engineers develop their works from this idea.*

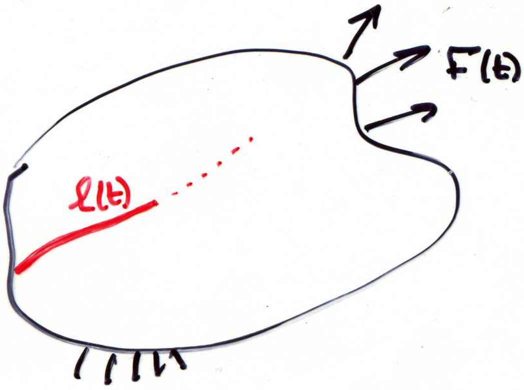
Unfortunately, the Griffith theory leads on a too simple assumption concerning the surface energy associated with a crack. With his assumption, we cannot account for various physical phenomena, like the initiation of cracks or the propagation of cracks by fatigue. To correct these drawbacks, we must change the form of the surface energy and introduce the concept of cohesive forces. On the other hand, several fundamental issues appear like the question of the irreversibility or the issue of the microcracking. In my seminar, I will try to present the basic ingredients of this variational theory of fracture, the most important results that we have obtained and also some still open problems.

Referenti dell'invito: Stefano Bennati, Aldo Frediani e Paolo S. Valvo.
I seminari saranno tenuti nella sala riunioni del DIS.

Pisa, 7 giugno 2006

Il Coordinatore del Corso di Dottorato
(*Prof. Stefano Bennati*)

GRIFFITH THEORY =



A surface energy assumption

G_c

+

A criterion of propagation

$$(G - G_c)\,\dot{\ell} = 0$$

$$\dot{\ell} \geq 0 \quad , \quad G \leq G_c$$

+ An irreversibility criterion

<u>Remark</u> Griffith law $\approx$ An Optimality condition for a restricted family of cracks

Potential Energy

$$E_p(\ell, t) = \min_{u \in \mathcal{C}(\ell, t)} \left\{ \frac{1}{2} \int_{\Omega_\ell} A\varepsilon(u)\cdot\varepsilon(u)\,dx - f_t(u) \right\}$$

Energy Release Rate

$$G = -\frac{\partial E_p}{\partial \ell}(\ell, t)$$

Total Energy

$$\mathcal{E}(\ell, t) = E_p(\ell, t) + G_c\,\ell$$

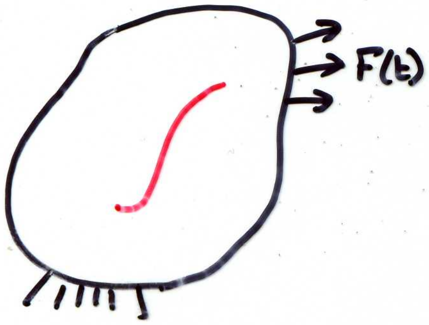
$\ell(t)$: "local minimum" $\mathcal{E}(\ell, t)$
(with irreversibility)

A variational approach of fracture

From GRIFFITH to PARIS

via BARENBLATT

Elasticity + "CRACKS"

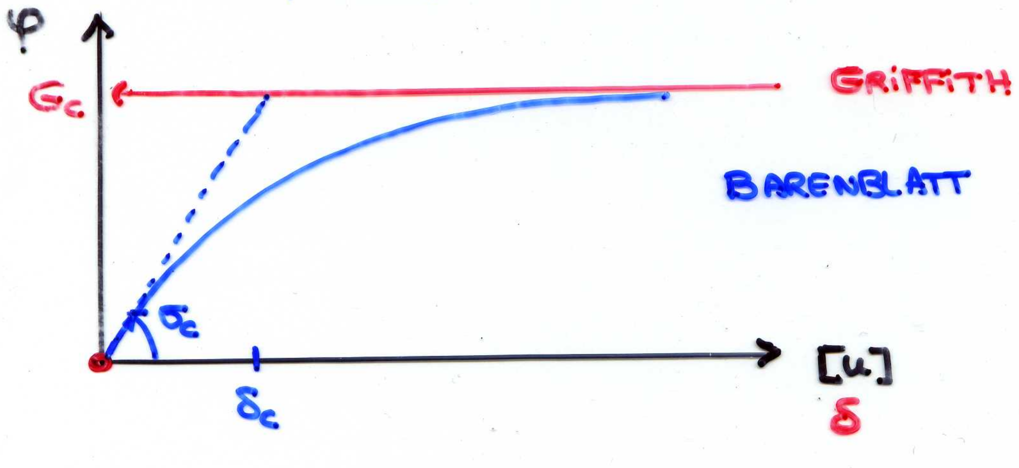


$u(t)$?

$S_{u(t)}$?

↑

(Global / Local) Minimization of the "total" energy



Some issues

- The INITIATION of CRACKS
- The CONCEPT of ~~PROCESS~~ ZONE
- The IRREVERSIBILITY of CRACK PROPAGATION
- The PHENOMENON of FATIGUE

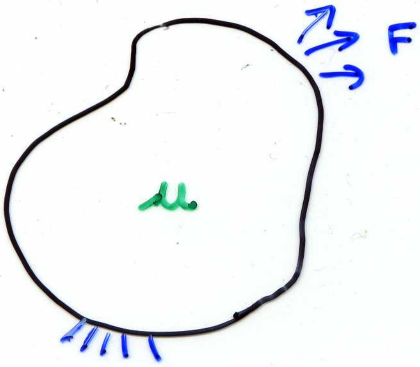
CONTEXT :

- LINEAR ELASTICITY – SMALL DISPLACEMENTS
- QUASI-STATIC – RATE INDEPENDENT PROCESSES
- DYNAMICS

J.J. MARIGO

Initiation of crack with GRIFFITH theory ?

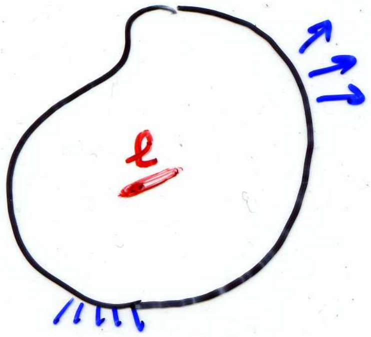
2D



convex "small" strains

$$u \quad \min_{v} \left\{ \int_{\Omega} W(\varepsilon(v))\,dx - F(v) \right\}$$

$S_u \,,\, S_v = \phi$



$$u_\ell \quad \min_{v} \left\{ \int_{\Omega_\ell} W(\varepsilon(v))\,dx - F(v) \right\}$$

$S_{u_\ell} \,,\, S_v \subset$ "crack ℓ"

$$E(u) \overset{?}{\underset{\geq}{\leq}} E(u_\ell) \qquad \text{For small } \ell$$

$$E(u) = \underbrace{\int_{\Omega} W(\varepsilon(u))\,dx - F(u)}_{\mathcal{P}_0}$$

$$E(u_\ell) = \underbrace{\int_{\Omega_\ell} W(\varepsilon(u_\ell))\,dx - F(u_\ell)}_{\mathcal{P}_\ell} + G_c\,\ell$$

In general $\quad \mathcal{P}_\ell = \mathcal{P}_0$ [illegible] $o(\ell)$, $\lim \dfrac{o(\ell)}{\ell} = 0.$

<u>Conclusion</u> The release of potential energy is <u>too small</u> in comparison to <u>surface energy</u>

$\Downarrow$

The <u>elastic response</u> is a <u>local minimum</u> , $\forall F$

Initiation of cracks with BARENBLATT theory!

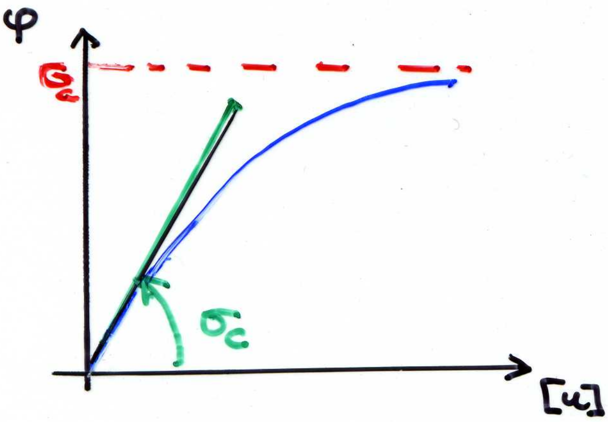
GRIFFITH defects

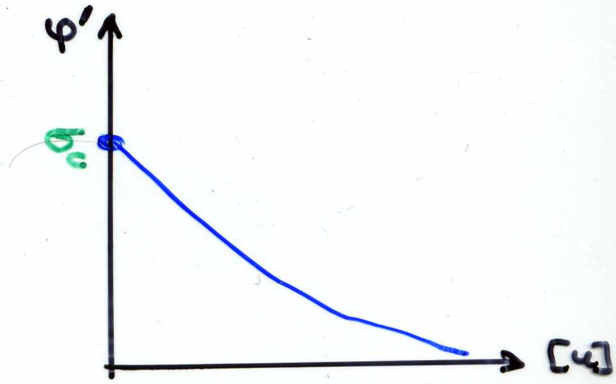
stress singularity

smooth crack opening

BARENBLATT corrections

Variational approach (Del Piero)





$$E_s(u) = \int_{S_u} \varphi([u])\, dS$$

$$\min_{u} \left\{ E_b(u) + E_s(u) - W_e(u) \right\}$$

Barenblatt energy

+

Local Minimum

$\Rightarrow$

Equilibrium

+

Cohesive Forces

+

$\boxed{\text{stress yield}}$

1st order

+

stability

2nd order

$$E(u+hv) = E(u) + h\,F(u;v) + \frac{h^2}{2}\,G(u,v) + \dots$$

1st order $\quad F(u;v) \geqslant 0 \quad \forall v$ "admissible"

2nd order $\quad G(u,v) \geqslant 0 \quad \forall v$ "admissible"

$F(u,v) = 0$

$$\boxed{\begin{array}{lcl} \bullet \text{ Energie de Barenblatt} & & \bullet \text{ Equilibre} \\ + & \Rightarrow & + \\ \bullet \text{ Minimum local} & & \bullet \text{ Seuil en contrainte} \end{array}}$$

. **Preuve**. (Hypothèse : 1D ; $[u] \geq 0$)

$$(*) \quad 0 \leq DE(u)(v) = \int_{\Omega \setminus S(u) \cup S(v)} \overset{Au'}{\overset{\shortparallel}{\sigma}} v' \, dx + \sum_{S(u) \cup S(v)} \varphi'([u])[v] - f(v)$$

$\forall v$ admissible.

(1) $\underline{S(v) = \phi}$ $\rightarrow$ $\boxed{\begin{array}{ll} \sigma' + f = 0 & \text{dans } \Omega \setminus S(u) \\ \sigma = F & \text{sur } \partial_F \Omega \end{array}}$

Equilibre

$$(*) \quad 0 \leq \sum_{S(u) \cup S(v)} (\varphi'([u]) - \sigma)[v] \quad , \forall v \text{ admissible}$$

(2) $\underline{S(v) \subset S(u)}$ $(\Rightarrow [v]$ arbitraire sur $S(u))$

$\rightarrow$ $\boxed{\sigma = \varphi'([u]) \quad \text{sur } S(u)}$

Forces de cohésion

$$(*) \quad 0 \leq \sum_{S(v) \setminus S(u)} (\varphi'(0) - \sigma)[v] \quad , \forall v \text{ admissible}$$

(3) $\underline{S(v) \setminus S(u) = \{x_0\}}$ $(\Rightarrow [v](x_0) > 0)$

$\rightarrow$ $\boxed{\sigma \leq \varphi'(0) = \sigma_c \quad \text{dans } \Omega \setminus S(u)}$

Seuil en contrainte

In 3D

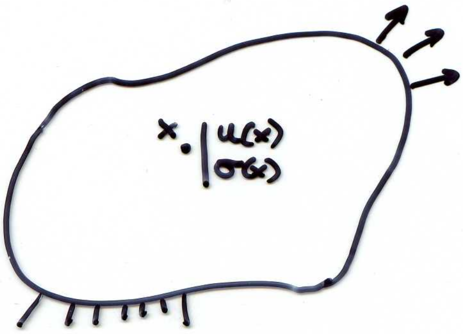
- H1. Isotropic material

$$\varphi(\delta, n) = \phi(\delta\cdot n, \|\delta - \delta\cdot n\, n\|)$$

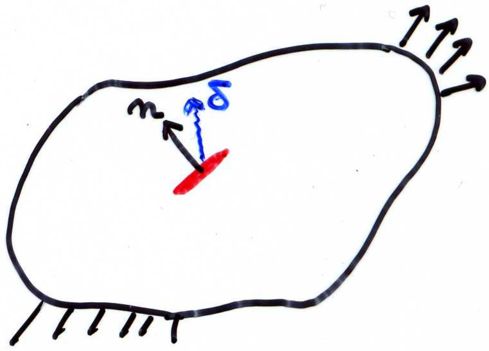
- H2. ϕ smooth, $\phi(0,0)=0$

$$\lim_{h\to 0^+} \frac{1}{h}\phi(h\alpha, h\beta) = \sigma_c\,\alpha + \tau_c\,\beta$$

- H3. Non interpenetration : $\delta\cdot n \geq 0$



Elastic response



$$\mathcal{E}(u) \leq \mathcal{E}(u + h\,w)$$

$$\mathcal{E}(v) = \frac{1}{2}\int_{\Omega\setminus S_v} A\,\varepsilon(v)\cdot\varepsilon(v)\,dx + \int_{S_v}\varphi([v], n_v)\,dS - (f, v)$$

$$\Downarrow$$

$$\sigma(x)\,n\cdot\delta \leq \sigma_c\,\delta\cdot n + \tau_c\,\|\delta - \delta\cdot n\,n\| \quad \forall x, \forall n, \forall \delta$$

$$\Updownarrow$$

$\sup_i \sigma_i \leq \sigma_c$ Maximal traction	$\sup_{i,j} \sigma_i - \sigma_j \leq \tau_c$ Maximal shear

Case 1 : Φ differentiable at (0,0)

$$\phi'(\alpha, \beta) = \sigma_c \alpha + \tau_c \beta$$

σ_c, τ_c: material constants (>0)

$$\sigma n \cdot \delta \leq \sigma_c \, \delta \cdot n + \tau_c \| \delta - \delta \cdot n \, n \|$$

$\forall n, \delta \; ; \quad \delta \cdot n \geq 0$

$$\Updownarrow$$

$$\boxed{\sup_i \sigma_i \leq \sigma_c} \quad \text{and} \quad \boxed{\sup_{i,j} \{\sigma_i - \sigma_j\} \leq \tau_c}$$

Maximal Traction — Maximal Shear

Remark Independent of the form of Φ (except isotropy + regularity)

Case 2 : Φ admits directional derivatives

- $\phi'(\alpha, \beta)$ positively homogeneous of degree 1 in (α,β)
- $\alpha \geq 0$, $\beta \geq 0$

$$\circledast \qquad \sigma n \cdot \delta \leq \phi'(\delta . n \, , \, \| \delta - \delta n \, n \|) \quad , \quad \delta . n \geq 0$$

(i) $\delta \parallel n$

$$\circledast \Rightarrow \quad \sigma n . n \leq \phi'(1,0) = \sigma_c \quad , \forall n : \|n\| = 1$$

$$\Rightarrow \quad \sup_i \sigma_i \leq \sigma_c$$

(ii) $\delta = \alpha n + t$, $t . n = 0$, $\alpha \geq 0$.

$$\circledast \Rightarrow \quad |\sigma n . t| \leq \phi'(\alpha, 1) - \alpha \, \sigma n . n \quad , \begin{array}{l} \forall \alpha \geq 0 \\ \forall n : \|n\| = 1 \end{array}$$

$$\Rightarrow \quad |\sigma n . t| \leq - \sup_{\alpha \geq 0} \{ \alpha \, \sigma n . n - \phi'(\alpha, 1) \}$$

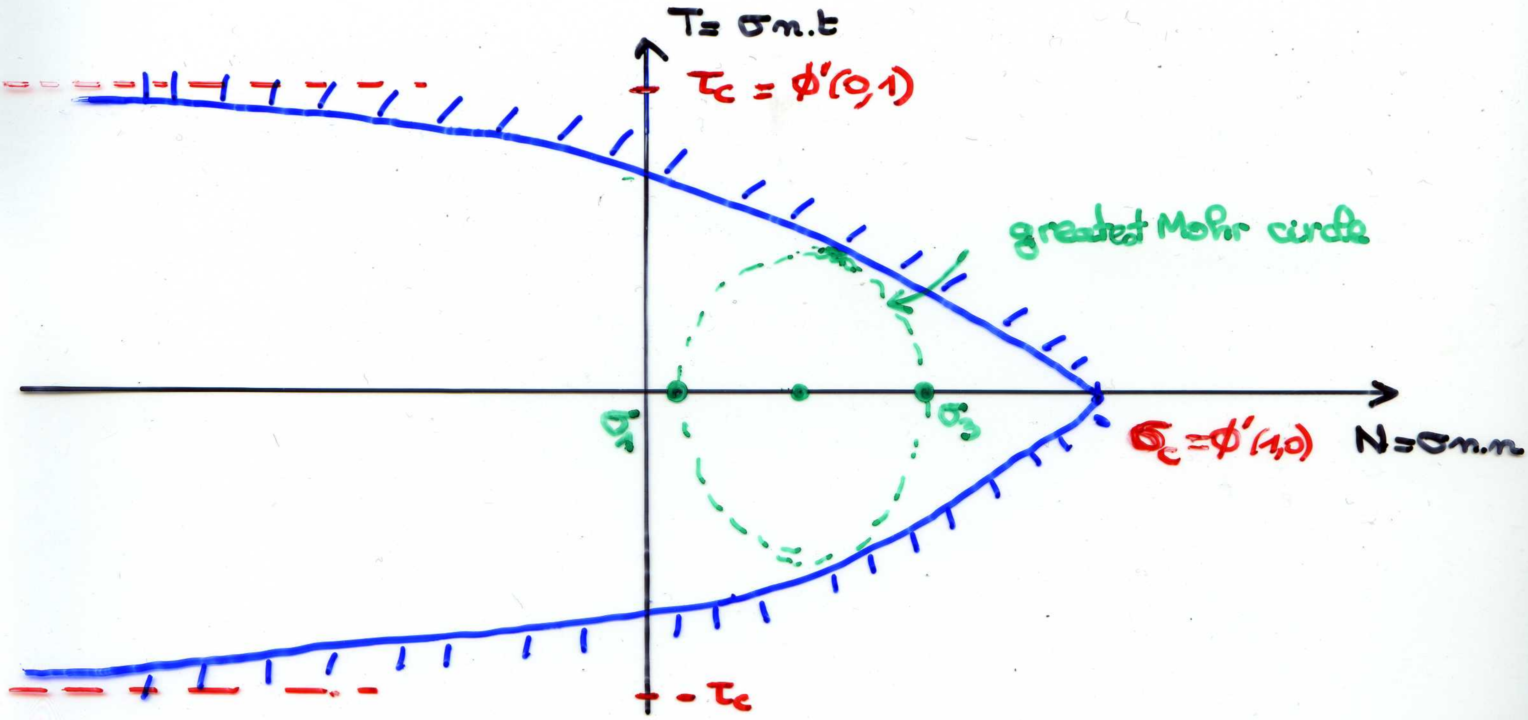
$$= - \psi^*(\sigma n . n)$$

$\uparrow$

Legendre transform of $\phi'(\cdot, 1) + I_{\mathbb{R}^+}$

Intrinsic curve

$$|\sigma n . t| + \psi^*(\sigma n . n) \leq 0 \qquad \forall n, t : n.t = 0$$



<u>Conclusion</u>

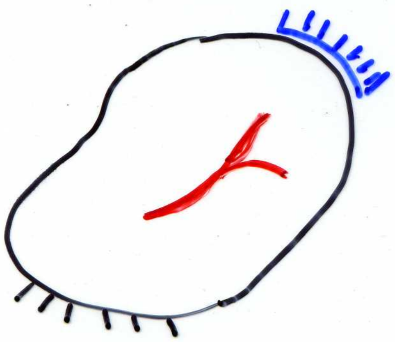
$$\boxed{\;|\sigma_3 - \sigma_1| + 2\,\Psi_*\!\left(\frac{\sigma_1+\sigma_3}{2}\right) \leq 0\;}$$

$$\Psi_*(s) = \sup_{\omega\in[0,\pi/2]} \left\{ s\cos\omega - \phi'(\cos\omega, \sin\omega) \right\}$$

Envelop of greatest Mohr circles

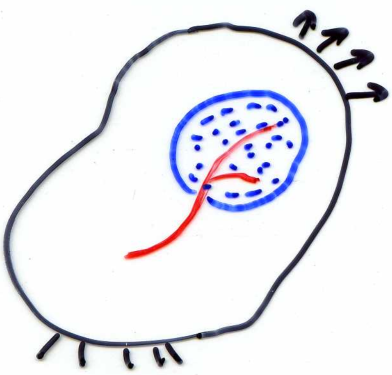
The CONCEPT of PROCESS ZONE
=
Relaxation of the energy

1. With GRIFFITH



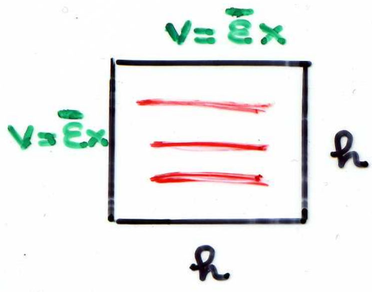
Existence of the minimum
=
No PROCESS ZONE

2- with BARENBLATT



No existence of the minimum
=
PROCESS ZONE
=
FINE MIXTURE
of
Elastic deformation
and
MICRO JUMPS

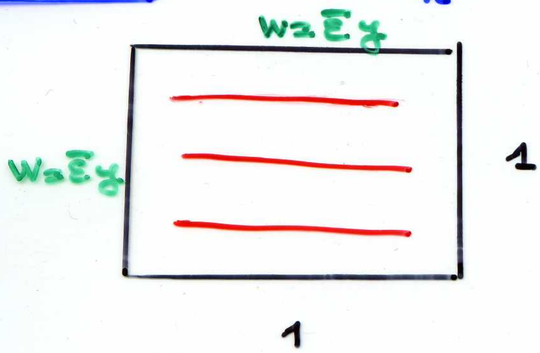
Effective bulk energy



Cube Q_h

$$\mathrm{vol}(Q_h)\, W_h^{eff}(\bar{E}) = \underset{\substack{v = \bar{E}x \\ \text{on} \\ \partial Q_h}}{\mathrm{Min}} \left\{ \int_{Q_h \setminus S_v} W(\varepsilon(v))\,dx + \int_{S_v} \varphi([v])\,dS \right\}$$

Rescaling $y = \frac{x}{h}$, $w(y) = \frac{v(x)}{h}$



Cube Q

$$W_h^{eff}(\bar{E}) = \underset{\substack{w = \bar{E}y \\ \text{on} \\ \partial Q}}{\mathrm{Min}} \left\{ \int_{Q \setminus S_w} W(\varepsilon(w))\,dy + \int_{S_w} \frac{\varphi(h[w])}{h}\,dS \right\}$$

" At the limit " $h \to 0$

$$W^{eff}(\bar{E}) = \underset{\substack{w = \bar{E}y \\ \text{on} \\ \partial Q}}{\mathrm{Min}} \left\{ \int_{Q \setminus S_w} W(\varepsilon(w))\,dy + \int_{S_w} \phi'([w]n, \|\ldots\|)\,ds \right\}$$

Bouchitté et al. Definition

Lower bound

$$W^{eff}(\overline{\varepsilon}) = \inf_{w} \left\{ \int_{Q\setminus S_w} W(\varepsilon(w))\,dy + \int_{S_w} \Phi'(\dots) \right\}$$

$$= \inf_{w} \sup_{\tau} \left\{ \int_{Q\setminus S_w} (\tau \cdot \varepsilon(w) - \underbrace{W^*(\tau)}_{\text{Legendre transform}})\,dy + \int_{S_w} \Phi'(\dots) \right\}$$

$$\geq \sup_{\tau} \inf_{w} \left\{ \dots \right\}$$

$$\geq \sup_{\sigma \in \mathbb{M}^3_s} \left\{ \sigma \cdot \overline{\varepsilon} - W^*(\sigma) + \underbrace{\inf_{\substack{w:\\ w=0 \\ \text{on} \\ \partial Q}} \int_{S_w} (\Phi'(\dots) - \sigma n \cdot [w])\,ds}_{= -I_K(\sigma)} \right\}$$

I_K = Indicatrix function of Admissible stress convex set

$$\rightarrow \quad \boxed{W^{eff} \geq (W^* + I_K)^*}$$

Upper bound

: construction of a "good" family of w.

$$\rightarrow \quad \dots \leq \dots$$

The issue of irreversibility

- Introduction of a MEMORY variable
- Modification of the MINIMIZATION procedure

1. With GRIFFITH

"crack" $\Gamma(t) = \bigcup_{s \leq t} S_{u(s)} \supset S_{u(t)}$

- The memory variable $= \Gamma$
- The INCREMENTAL Problem

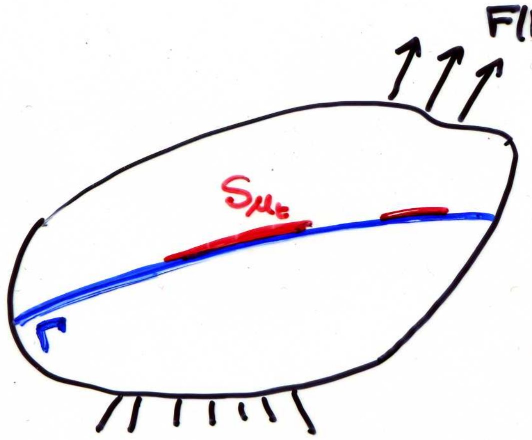
Γ_0 given

$$\begin{cases} u_i : \quad \underset{u \in \mathcal{C}_i}{\mathrm{Min}} \left\{ \int_{\Omega \setminus S_u} \frac{1}{2} A\varepsilon(u) \cdot \varepsilon(u)\, dx + G_c \mathcal{H}^2(S_u \setminus \Gamma_{i-1}) \right\} \\ \Gamma_i = \Gamma_{i-1} \cup S_{u_i} \end{cases}$$

- The "Continuous in time" problem (Francfort-Larsen, Dal Maso, ...)

$= \lim_{\Delta t \to 0}$ Incremental problem

2. With BARENBLATT (Restricted context : $S_u \subset \Gamma$ fixed)



Memory variables

Cumulated Opening $\quad \dot{\delta}_n = ([\dot{u}]\cdot n)^+$

Cumulated Sliding $\quad \dot{\delta}_T = \|[\dot{u}_T]\|$

Remark : Other choices ?

→ Density surface energy $\phi(\delta_n, \delta_T)$

The INCREMENTAL PROBLEM

- δ_n^0, δ_T^0, u^0 given
- u^i :
$$\min_{v \in \mathcal{C}^i} \left\{ \frac{1}{2} \int_{\Omega\setminus\Gamma} A\varepsilon(v).\varepsilon(v)\,dx + \int_\Gamma \phi\Big(\delta_n^{i-1} + ([v-u^{i-1}].n)^+ ,\ \|\delta_T^{i-1} - [v_T - u_T^{i-1}]\|\Big) ds - f^i(v) \right\}$$
- $\delta_n^i = \delta_n^{i-1} + ([u^i - u^{i-1}].n)^+$, $\quad \delta_T^i = \delta_T^{i-1} + \|[u_T^i - u_T^{i-1}]\|$

OPEN PROBLEMS : - Relaxation ?

- $\lim_{\Delta t \to 0}$?