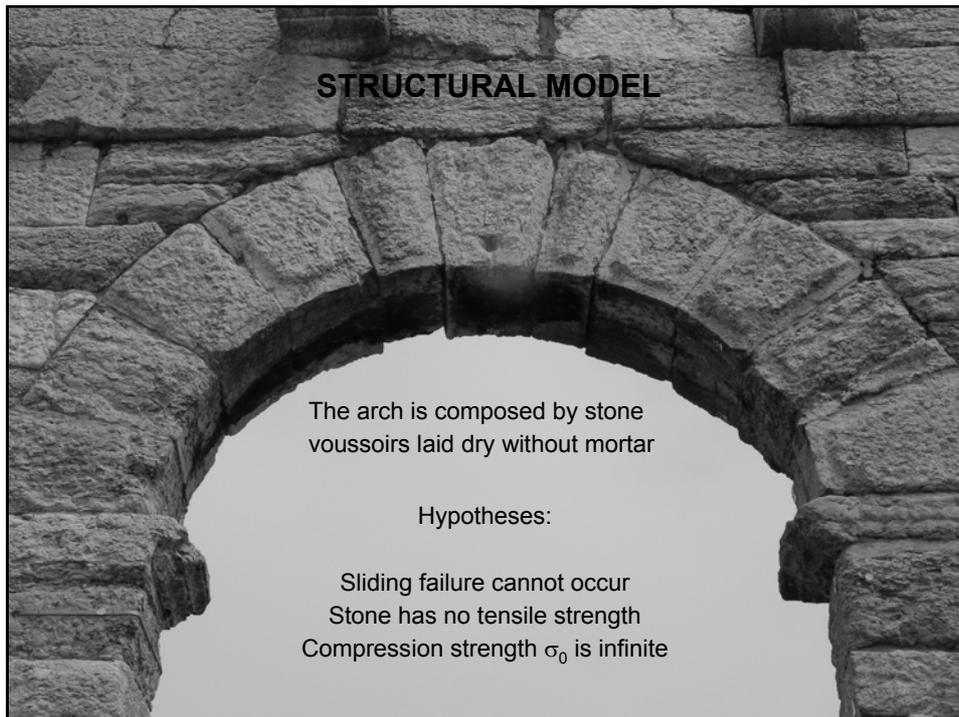**Università degli Studi di Pisa**
Facoltà di Ingegneria
Dottorato in Scienze e Tecniche dell'Ingegneria Civile
26 maggio 2006

LIMIT ANALYSIS OF MASONRY ARCH BRIDGES

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LIMIT ANALYSIS OF MASONRY ARCH BRIDGES

➢**Limit analysis**

➢**Funicular arch**

➢**Check under dead loads**

➢**Check under moving loads**

➢**Check under horizontal actions**

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STRUCTURAL MODEL

The arch is composed by stone voussoirs laid dry without mortar

Hypotheses:

Sliding failure cannot occur
Stone has no tensile strength
Compression strength σ_0 is infinite

LIMIT ANALYSIS: HYPOTHESES

Sliding failure cannot occur: in fact, friction between voussoirs is high enough to suppose that they cannot slide on one another. Besides, shear forces are low.

Stone has no tensile strength: this is almost exactly true in this kind of structures, made up of voussoir laid either dry or with very weak mortar: although stone itself may have some tensile strength, the joints will not, therefore no tensile forces can be transmitted from one voussoir to another.

The compression strength, σ_0, can be supposed to be infinite: as we can see from the analysis of existing bridges, stresses are low enough not to allow crushing of the material. This observation is equivalent to the assumption that stone has an infinite compressive strength.

NO-TENSION MATERIAL

Constitutive law
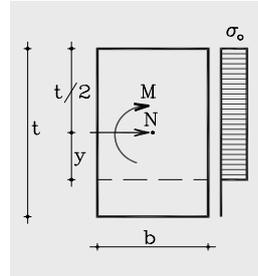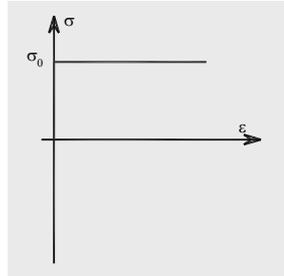
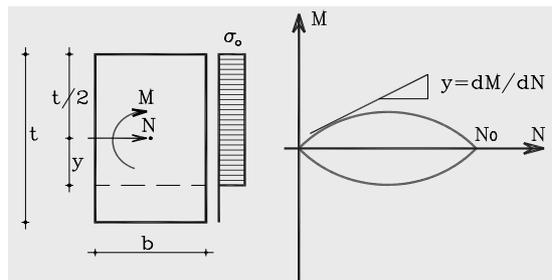For each "y" we calculate N and M:

$$N = by\,\sigma_0 \qquad\qquad M = by\sigma_0\left(\frac{t}{2} - y\right)$$

The limit compression force and limit moment are

$$N_0 = bt\,\sigma_0 \qquad\qquad M_0 = b\frac{t^2}{8}\sigma_0$$

LIMIT DOMAIN: Rectangular Cross-section

The limit domain is composed by two parabolic curves

$$M = \pm\frac{bt^2}{2}\,\sigma_0\,\frac{N}{N_0}\left(1 - \frac{N}{N_0}\right)$$

The position of the neutral axis is individualised by the tangent to the domain
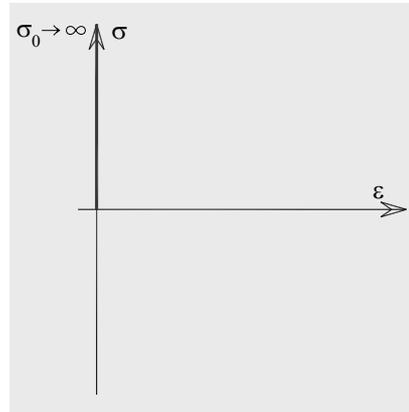
$$y = \frac{dM}{dN}$$

The curvature is constant $\qquad \dfrac{d^2M}{dN^2} = \mp\dfrac{1}{b\sigma_0}$
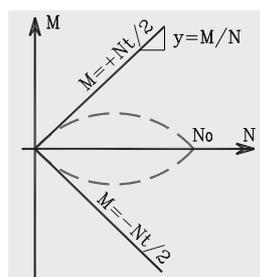
3

MATERIAL: Constitutive law

Compressive strength is infinite

No strain

No tensile strength

LIMIT DOMAIN for σ → ∞

If

$$\sigma_0 \to \infty$$

then the two parabolic curves become two straight lines, tangential to parabolic ones at the origin

$$M = \pm N\, t/2$$

Each point of these lines represents a state of stress in which the axial force N lies in one of the masonry surfaces, its distance from the centre-point of the section being given by

$$y = \frac{dM}{dN} = \pm \frac{t}{2} = \frac{M}{N} = \cos t$$
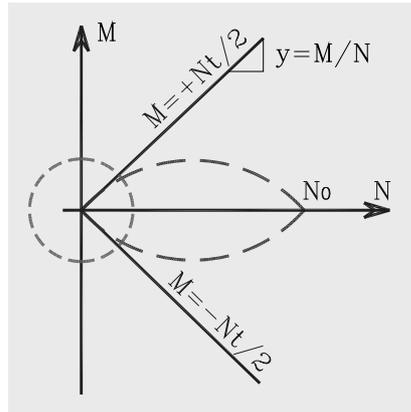
The curvature is

$$\frac{d^2M}{dN^2} = 0$$

LIMIT DOMAIN: Considerations

The hypothesis

$$\sigma_0 \to \infty$$

is equivalent to the hypothesis that the axial force N is very low in comparison with N_0:

$$N \ll N_0$$

So only the portion of the limit domain very close to the origin is meaningful

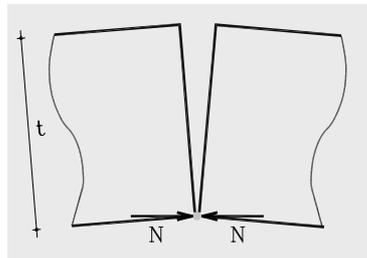In this zone the domain can be confused with the two tangential lines

COLLAPSE OF THE CROSS-SECTION

On the point of collapse the **resultant force** is at the extrados or at the intrados of the cross-section.

Then the collapse consists in the formation of a **hinge** at a free edge.

The two cross-sections can **rotate** one to the other around the extrados or the intrados.

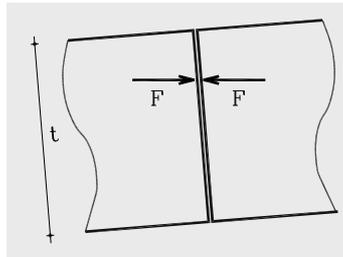At internal hinges the resultant must be **tangential** to the edge: F = N.

SERVICE CONDITION

Obviously, in service conditions, the **resultant force** must be at an internal point of the cross-section.

If no hypotheses are made on the constitutive relation-ship of the material the effective diagram of **stresses** cannot be determined.

COLLAPSE OF THE STRUCTURE

Failure of the arch occurs when **sufficient hinges** form to turn the structure into a mechanism
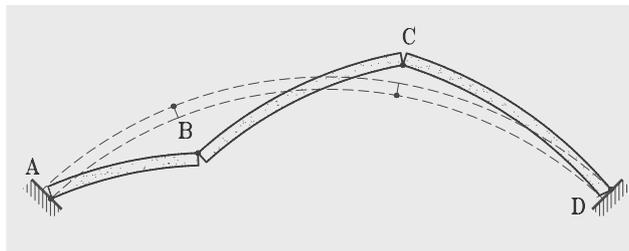

In other words the arch is **on the point of collapse** if we can find a line of thrust lying wholly within the masonry, which:

- represents an equilibrium state for the structure under the external loads and
- allows the formation of sufficient hinges to transform it into a mechanism.

UNIQUENESS THEOREM

To state the uniqueness theorem, it is necessary to point out a preliminary observation.

Since a proportional increasing load cannot cause any change in the line of the thrust, it is important to **distinguish** dead loads from variable loads.
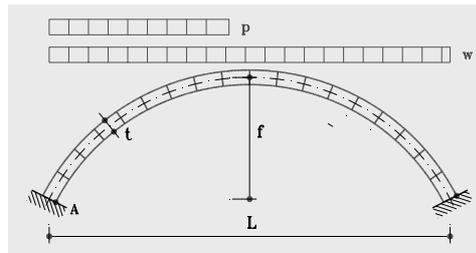

Uniqueness theorem

If the **variable loads** are specified as ratios of one of their number and they have been increased from their working values to the collapse values by a load factor, while the dead load does not change, the value of that load factor on the point of collapse is unique.

SAFE THEOREM

Safe theorem

A structure is safe if a line of thrust in equilibrium with the service loads and lying wholly within the masonry can be found
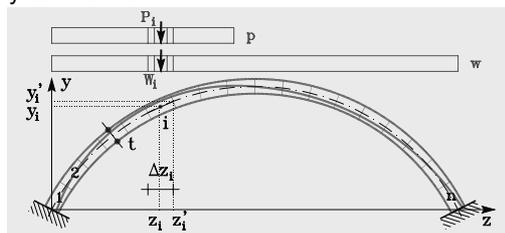
Considerations

The thrust line of the safe theorem need not be the **actual thrust line**: every thrust line in equilibrium with external loads, lying within the arch profile, if any, can be chosen to check the structure

We do not know the **actual stresses** in the cross-section. We did not make any assumption about the material constitutive relationship, but the fact that the thrust line lies within the masonry ensures that there is only compressive action, which can be transmitted from one cross-section to the next

LINE OF THRUST - 1

A line of thrust indicates a suitable position and slope of the resultant force at z_i

For a given external load, we can draw infinite lines of thrust. A line of thrust if defined if we fix 3 points (statically admissible)

If P_1, P_2 and P_3 are fixed, we can write the equilibrium equations
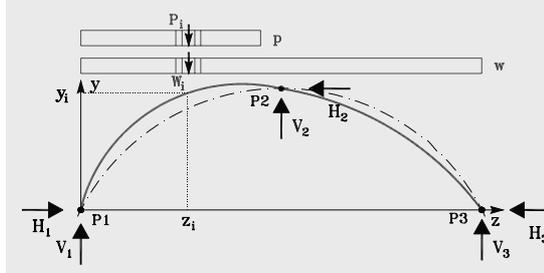
$$H_1 - H_3 = 0$$

$$\sum_1^n W_i + \lambda \sum_1^n P_i - V_1 - V_3 = 0$$

$$\sum_1^n W_i \cdot (z_3 - z_i) + \lambda \sum_1^n P_i (z_3 - z_i) - V_1 (z_3 - z_1) - H_1 (y_3 - y_1) = 0$$

$$\sum_1^{i_B} W_j \cdot (z_2 - z_j) + \lambda \sum_1^{i_B} P_j (z_2 - z_j) - V_1 (z_2 - z_1) + H_1 (y_2 - y_1) = 0$$

We deduce $\qquad V_1, V_2, H_1$

LINE OF THRUST - 2

Resultant forces at "z_i":

$$V_i = V_1 - \sum_1^i W_j - \sum_1^i P_j$$

$$H_i = H_1$$

$$M_i = V_1 \cdot (z_{ii} - z_1) - H_1 \cdot (y_i - y_1) - \sum_1^i W_j (z_i - z_j) - \sum_1^i P_j (z_i - z_j)$$

It must be

$$M_i = 0$$

We deduce the height of the resultant at "z_i":

$$(y_i - y_1) = \frac{1}{H_1} \left[V_1 \cdot (z_i - z_1) - \sum_1^i W_j (z_i - z_j) - \sum_1^i P_j (z_i - z_j) \right]$$
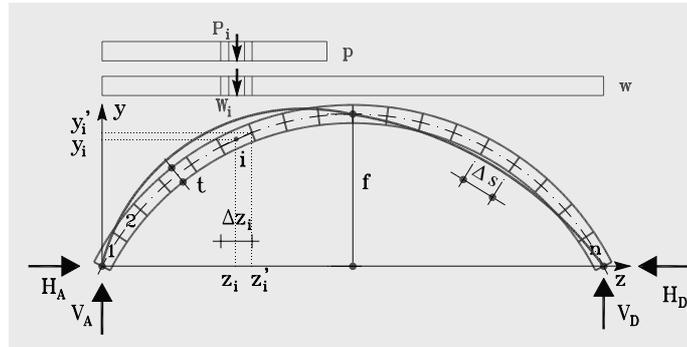
LINE OF THRUST - 3

We can find a suitable line of thrust in the arch by fixing three hinges.

If we can find a line of thrust lying wholly within the masonry, then the arch is safe.

If the centre line is a suitable line of thrust then the arch is funicular of the given load

LIMIT ANALYSIS OF MASONRY ARCH BRIDGES

> ➤ **Limit analysis**

> ➤ **Funicular arch**

> ➤ **Check under dead loads**

> ➤ **Check under moving loads**

> ➤ **Check under horizontal actions**

FUNICULAR ARCH

The arch is funicular with reference to a **load condition**

The arch is funicular with reference to a load condition if so is its **centre line**

Usually, we consider arches funicular with reference to **dead loads**

Suppose dead load to be **uniformly distributed** along the span, then the centre line of the funicular arch has **parabolic shape**

We consider **moving load coming on to the bridge**.

So the load condition consists in a uniformly distributed load between **z=0** and the end load abscissa **z_p**, which individualised the load condition

FUNICULAR ARCH: Collapse Mechanism

Under a fixed load condition, for any **given mechanism**, and so for any field of **velocity η**, consistent with the external constraints, which satisfies the compatibility condition:

$$z_B < z_2 < z_C$$
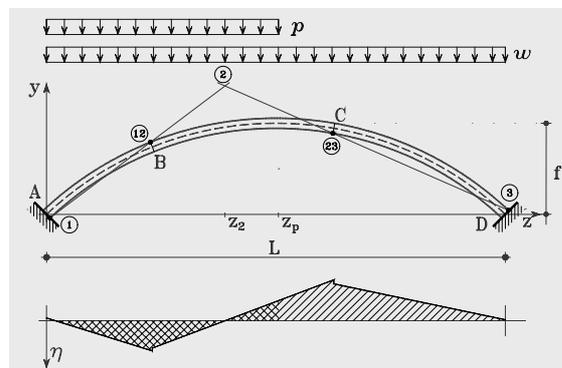
we calculate the **virtual powers**:

$$P_w = w \cdot \int_0^L \eta \; dz$$

$$P_p = p \cdot \int_0^{z_p} \eta \; dz$$

The **equilibrium equation** can be written

$$P_w + \lambda \, P_p = 0$$

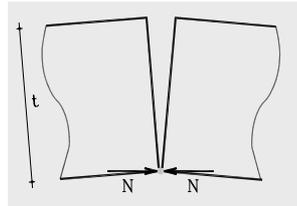The **kinematically admissible** load factor can be deduced

$$\lambda = -P_w / P_p$$

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FUNICULAR ARCH: Equilibrium Equation

We must find the minimum **kinematically admissible** load factor, which is also statically admissible:

load factor = min (kinematic admissible λ) = max (statically admissible λ)

This can be done by using an **iteration procedure**.

It is very important to observe that in the equilibrium equation

$$P_w + \lambda P_p = 0$$

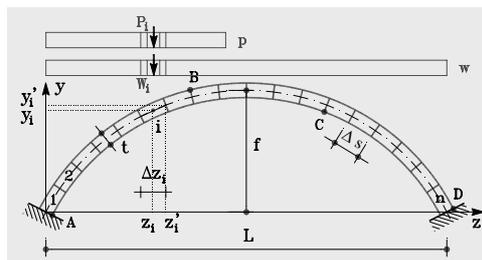there are no terms with the internal forces acting at the hinges.

This circumstance is greatly appreciable, because the values of the internal forces are still unknown

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ITERATION PROCEDURE - 1

Dead and travelling loads for each voussoirs

$$W_i = (w) \cdot \Delta z_i$$

$$P_i = p \cdot \Delta z_i$$

Virtual powers:

$$P_w = \sum_1^n W_i \cdot \eta_i \qquad\qquad P_p = \sum_1^n P_i \cdot \eta_i$$

Kinematically admissible load factor:

$$\lambda = -\frac{\sum_1^n W_i \cdot \eta_i}{\sum_1^n P_i \cdot \eta_i}$$

ITERATION PROCEDURE - 2

Loads: $w + \lambda p$
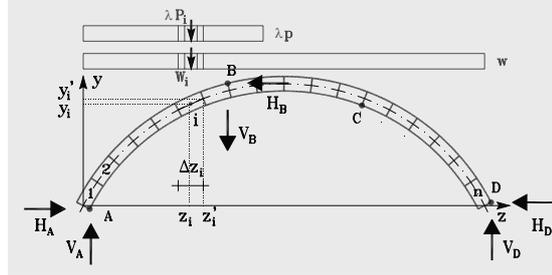

Equilibrium equations

$$H_A - H_D = 0$$

$$\sum_1^n W_i + \lambda \sum_1^n P_i - V_A - V_D = 0$$

$$\sum_1^n W_i \cdot (z_D - z_i) + \lambda \sum_1^n P_i (z_D - z_i) - V_A (z_D - z_A) - H_A (y_D - y_A) = 0$$

$$\sum_1^{i_B} W_j \cdot (z_B - z_j) + \lambda \sum_1^{i_B} P_j (z_B - z_j) - V_A (z_B - z_A) + H_A (y_B - y_A) = 0$$

We deduce $V_A,\ V_D,\ H_A$

ITERATION PROCEDURE - 3

Internal forces at section "i":

$$V_i = V_A - \sum_1^i W_j - \lambda \sum_1^i P_j$$

$$H_i = H_A$$

Finally

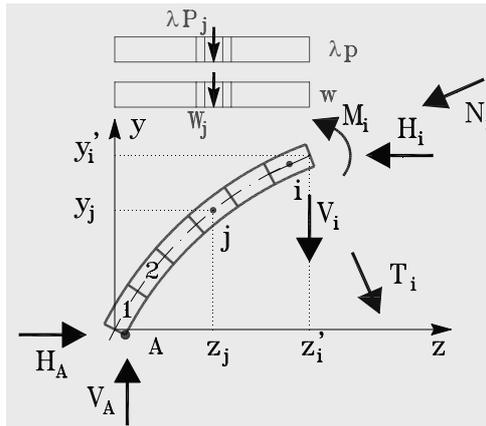$$T_i = V_i \cdot \cos\alpha_i - H_i \cdot \mathrm{sen}\alpha_i$$

$$N_i = V_i \cdot \mathrm{sen}\alpha_i + H_i \cdot \cos\alpha_i$$

$$M_i = V_A \cdot (z_i' - z_A) - H_A \cdot (y_i' - y_A) - \sum_1^i W_j (z_i' - z_j) - \lambda \sum_1^i P_j (z_i' - z_j)$$

We calculate the position of the resultant with reference to the centre line

$$e_i = M_i / N_i$$

FUNICUL ARCH: Iteration Procedure

1) Give a **first mechanism** and the corresponding diagram of velocity

2) The **equilibrium equation** can be written: $P_w + \lambda * P_p = 0$

3) The **kinematically admissible load factor** can be evaluated: ($\lambda = P_w/P_p$) and then the actual collapse travelling load (λp)

4) The **external reactions** can be found out

5) The **internal reaction** (N_i and M_i) can be evaluated (i.e., the line of thrust passing through the assumed hinges)

6) The **eccentricity** can be evaluated for each section: $e_i = M_i/N_i$

The value of λ is the **collapse load factor** only if

$$e_i \le t/2$$

at each section (i.e., the line of thrust passing through the assumed hinges is elsewhere within the masonry).
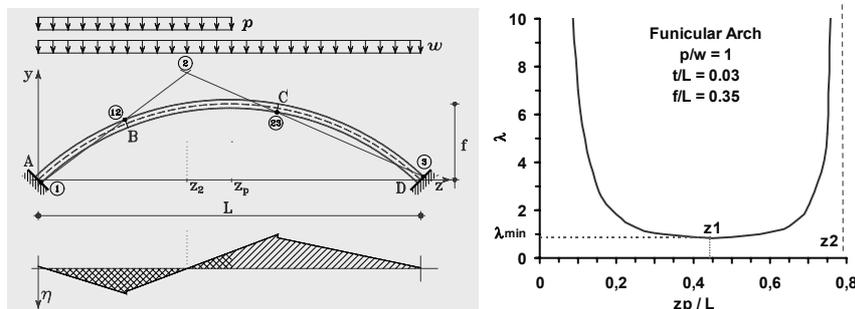
If it is not, in the next step we must **move** the hinges to the cross-section where there are the maximum distances between the line of thrust and the arch profile

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FUNICULAR ARCH: Numerical investigation

In the numerical investigation we assume   **p = w**
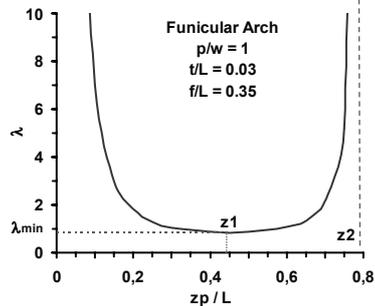
For each moving load condition (z_p) we find the collapse mechanism and the corresponding load factor

The minimum load factor is the collapse load factor λ_{min}

The diagrams shows two characteristic points

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# FUNICULAR ARCH: z_1

Funicular Arch
p/w = 1
t/L = 0.03
f/L = 0.35

The first one, z_1, individualises the minimum value of λ, which corresponds to the half-span load condition (z_p=0.45*L)

The small difference between 0.5 and the found value of z_1 is due to the offset of hinges from the centre line, which results in a non perfectly anti-symmetric mechanism of collapse with respect to the mid-span

Diagrams show a flat central part around that point, which leads to state that a load condition close to the worst is achieved for a wide range of load length

FUNICULAR ARCH: z_2
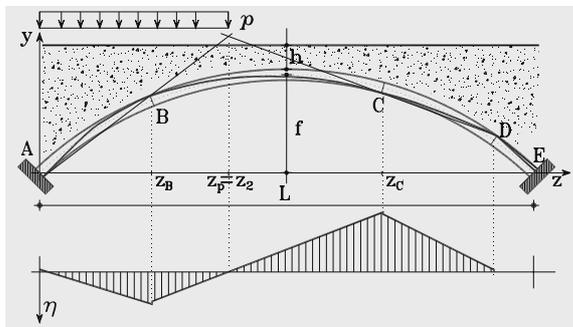

Funicular Arch
p/w = 1
t/L = 0.03
f/L = 0.35
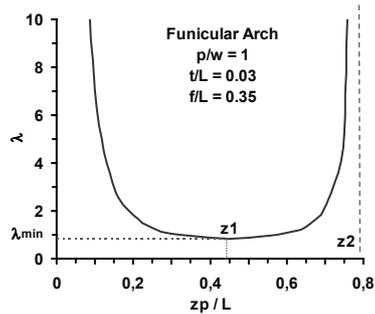
The diagram also shows the presence of a vertical asymptote at $z_p = z_2 \approx 0.70$

When the load length approaches L the stability of the arch increases

If $z_p > z_2$ there is no danger of failure, because the hypothesized mechanism cannot be exploited under such a load condition

It is worth reminding that the centre line of a parabolic arch coincides with the funicular polygon of a uniformly distributed load
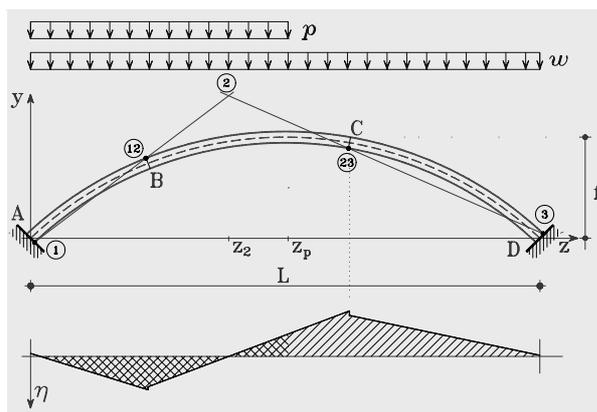
FUNICULAR ARCH: Minimum Load Factor

When $\quad \lambda = \lambda_{min}$

then $\qquad z_p = z_1$

it is also $\quad z_p = z_2$

Actually, for a given mechanism, this is the load condition for which P_p assumes its maximum value and λ is the minimum one

FUNICULAR ARCH: Dead Load Contribution

If
$$z_p = z_2$$
then
$$P_w = w \cdot \left(A^+ - A^- \right)$$
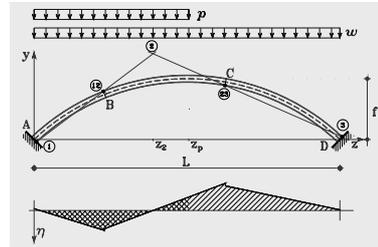$$P_p = p \cdot A^+$$

And the load factor is
$$\lambda = -\frac{w}{p} \cdot \left(1 - \frac{A^-}{A^+} \right)$$

In order λ to be positive, it must be $\qquad A^- > A^+$

On the point of collapse, the equilibrium is guaranteed by P_w, which must be negative. Dead loads have a stabilizing effect

FUNICULAR ARCH: z_C, z_2 and z_B versus z_p
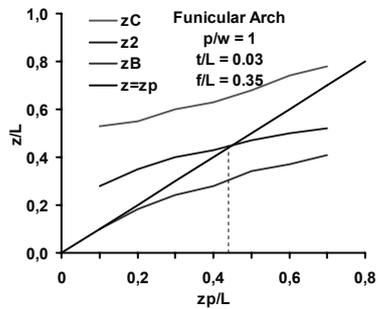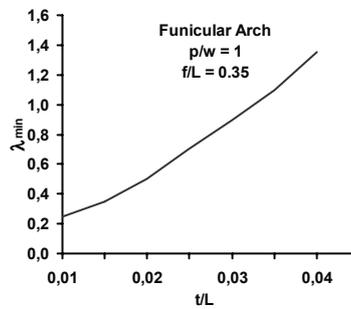

It is always $z_B < z_2 < z_C$

If $z_p = z_1$ it is also $z_1 = z_2$

FUNICULAR ARCH: z_1, z_2 versus t/L e f/L
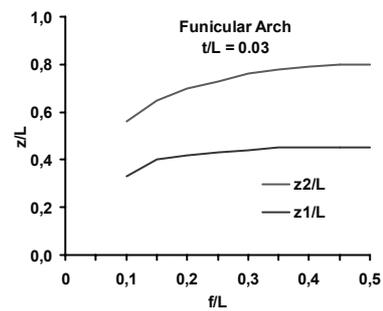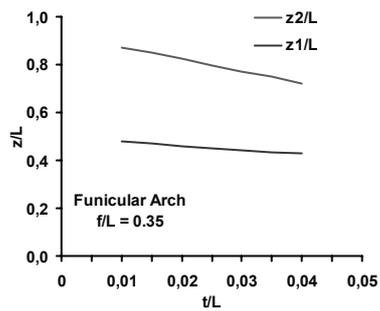
z_1 is independent of thickness t and of sag f. We assume $z_1 = 0.45*L$

z_2 shows higher variation

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FUNICULAR ARCH: λ_{min} versus t

Funicular Arch
p/w = 1
f/L = 0.35

As we expected, λ increases with t.

The relationship between λ and t is slightly different from a linear law

FUNICULAR ARCH: λ_{min} versus f
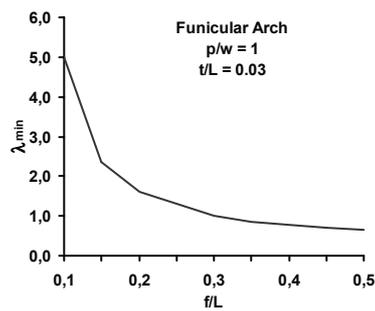
Funicular Arch
p/w = 1
t/L = 0.03

The load factor λ increases quickly as f decreases.

The reason for this behaviour is the following: when the arch has a small height, stresses due to dead loads are very high, so that the structure becomes less vulnerable to the travelling load. A very high load is necessary to move the line of thrust

FUNICULAR ARCH: influence of f
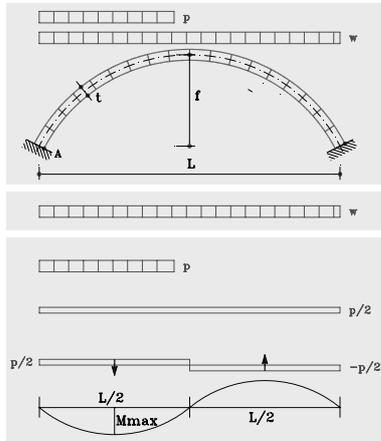
$$H_w = \frac{w \cdot L^2}{8 \cdot f} \qquad\qquad M_w = 0$$

$$H_p = \frac{p/2 \cdot L^2}{8 \cdot f} = \frac{p \cdot L^2}{16 \cdot f} \qquad\qquad M = 0$$

$$H = 0 \qquad\qquad M_p = \frac{p/2 \cdot (L/2)^2}{8} = \frac{p \cdot L^2}{64}$$

The eccentricity under p is related to

$$\frac{M_p}{H_p} = \frac{f}{4}$$

The eccentricity under w+p is related to

$$\frac{M_p}{H_w + H_p} = \frac{f}{4} \cdot \frac{p/w}{2 + p/w}$$

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DEAD LOADS: Minimum Thickness

If the centre-line of the voussoir coincides exactly with one line of thrust in equilibrium with dead loads (self-weight and fill), there is no danger of "mechanism-type" failure and the arch can be designed as thin as the material strength allows.
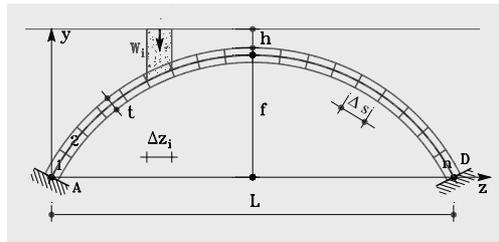

Otherwise a **minimum depth** of the arch exists to contain the line of thrust.

If the arch is thinner, then it will immediately collapse; if the arch thickness is larger than the minimum necessary to contain the thrust line, then it will stand.

MINIMUM THICKNESS

The **minimum depth** can be defined as follows: by supposing that the depth is exactly equal to the minimum one, then just one line of the thrust, lying wholly within the masonry, can be found. This line touches the arch profile, alternatively at the intrados and the extrados, at a sufficient number of points to turn the structure into a mechanism.
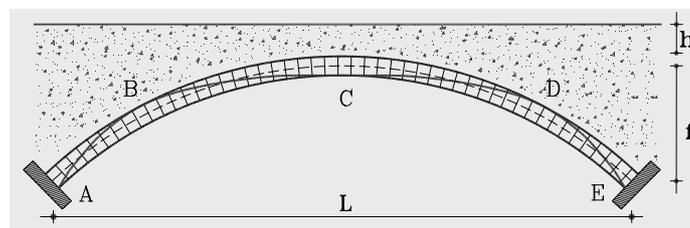
When this happens the arch is on the point of collapse under dead loads.

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DEAD LOADS: Collapse Mechanism

Since the structural geometry and loads are symmetric, the mechanism of collapse must be symmetric too. Consequently at least 5 hinges form, one of them being at the crown.
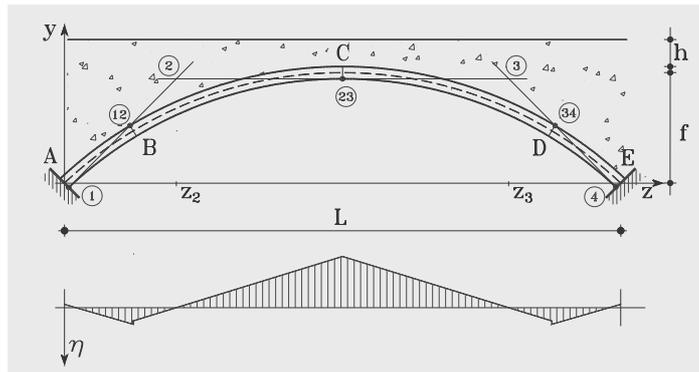
MINIMUM THICKNESS: PARABOLIC ARCH - 1

If we consider, with reference to a parabolic arch, a line of thrust in equilibrium with dead loads which touches the intrados at the crown and at the springings, we find that it is wholly above the intrados
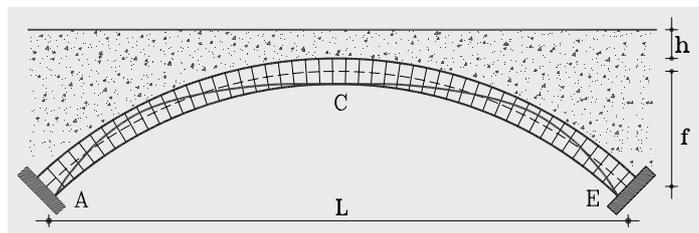
This consideration suggests how we can find the mechanism of collapse using an **iteration procedure** which can be started by using a given value of the depth and drawing the line of thrust passing through the intrados at the crown and the springings. The **maximum distance** between the intrados and the thrust line can be assumed as the value of the depth in the next step

MINIMUM THICKNESS: PARABOLIC ARCH - 2

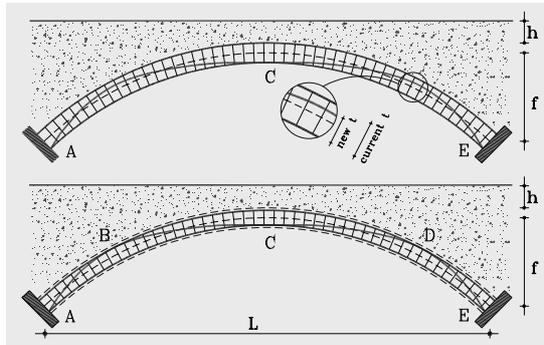We will find the minimum thickness when the corresponding line of thrust, passing through the three above mentioned points, lies wholly within the masonry and touches the extrados at two symmetrical points

Dead loads change in each iteration due to the arch thickness variation; as a consequence the thrust line changes too. Of course the line of thrust will be tangential to the arch profile at the internal hinges (B,C and D)

MINIMUM THICKNESS: CIRCULAR ARCH - 1

The assumption that a line of thrust touching the intrados at the crown and at the springings is wholly above the intrados is not always true in the case of a circular arch. So the mechanism of collapse may be different from the one described for the parabolic arch

For circular arch we prefer to define the geometry by means of the radius R and the angle of embrace β.

Consider the arch of fixed radius R and the series of arches obtained by increasing β
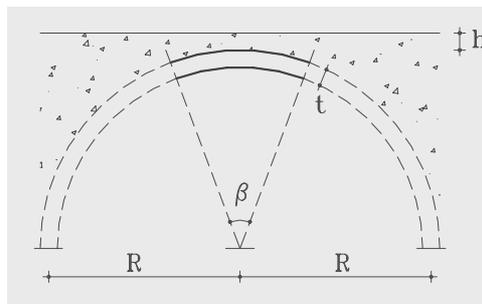
MINIMUM THICKNESS: CIRCULAR ARCH - 2

When β is small (f/L small), we can confirm what already said about the parabolic arch: a line of thrust in equilibrium with dead loads, which touches the intrados at the crown and at the springings, is wholly above the intrados
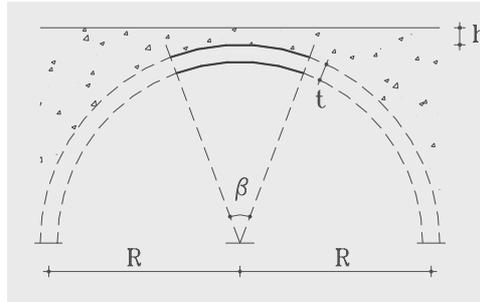

So the collapse mechanism is of the same kind as for the parabolic arch and we can use the same iteration procedure used for the parabolic arch

This happens up to a certain value β_1

MINIMUM THICKNESS: CIRCULAR ARCH - 3

If $\beta = \beta_1$, the collapse mechanism is still the same and the thrust line is tangential to the intrados at the springings

If β is greater than a particular value β_1, hinges A and E will not form at the springings and the mechanism will regard only the portion of the structure, individualised by $\beta = \beta_1$
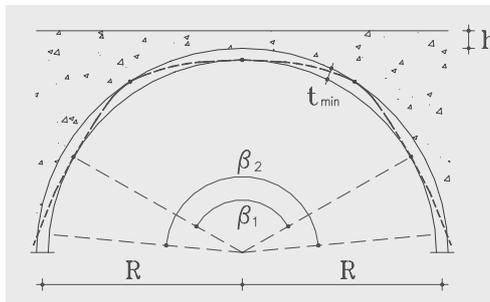
The thrust line is tangential to the intrados at $\beta = \beta_1$ and is internal to the arch profile in the sections between $\beta = \beta_1$ and the springings

The minimum thickness keep constant and equal to that relative to $\beta = \beta_1$

MINIMUM THICKNESS: CIRCULAR ARCH - 4

This is true up to a certain value β_2

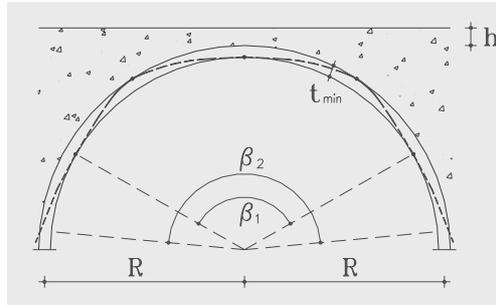For $\beta = \beta_2$ the line of thrust crosses the extrados at the springings

For $\beta > \beta_2$ the 5 hinges still individualize a part of the structure for which $\beta = \beta_1$ but the line of thrust crosses the extrados at two sections between β_1 and the springings

MINIMUM THICKNESS: CIRCULAR ARCH - 5

In order to determine the minimum thickness beyond this threshold, one should refer to the condition that the thrust line be within the arch profile near the springing, since there is no more question of mechanism failure
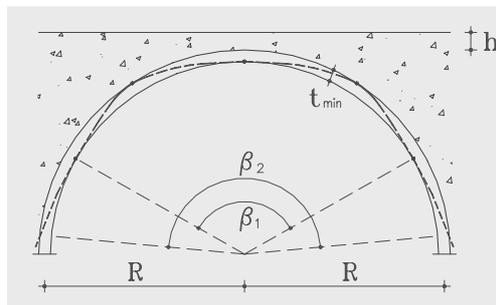
Anyway, in order to contain the thrust line in the arch profile, large increment of the thickness is needed

The values of β_1 and β_2 depend on the loads (dead load and backfill)

MINIMUM THICKNESS: CIRCULAR ARCH - 6

We deduce that the semicircular arch can be realised only with very high depth

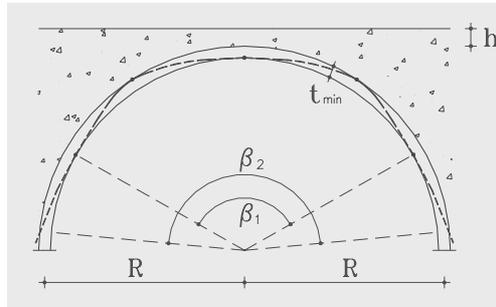The semicircular arch cannot be funicular of any vertical load condition. In fact the funicular curve of any vertical load, which is similar to the diagram of the bending moment of a simple supported beam, cannot present vertical tangent at the springings. In this case the shear force at the ends should be infinite

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MINIMUM THICKNESS: CIRCULAR ARCH - 7

The collapse mechanism may also change when *γ is very low*

In this case the thrust line touching the extrados at the crown and the springings is wholly below the extrados

So the iteration procedure can be started by using a given value of the depth and drawing the line of thrust passing through the extrados at crown and springings

The maximum distance between the extrados and the thrust line approximates the minimum thickness very well

However, it can be assumed as a trial value in the next step

As you can see the kind of collapse mechanism depends on the geometrical characteristics and on the loads

MINIMUM THICKNESS: CIRCULAR ARCH - 8
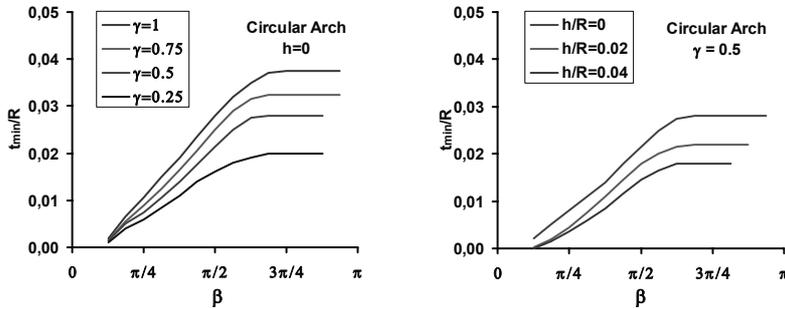
The diagrams of t_{min}/R versus β for different values of γ and h/R show the same intervals of β already individualised

The geometrical and load parameters have low influence on both β_1 and β_2, which can be assumed equal to:

$$\beta_1 = 2\pi/3 \qquad\qquad \beta_2 = 10\pi/11$$

γ = ratio between the weight per unit volume of the fill and of the masonry

COMPARISON BETWEEN PARABOLIC AND CIRCULAR ARCHES - 1

In order to compare parabolic and circular arches, the geometry of this latest must be defined by means of f/L and the height above the crown by means of h/L, as for the parabolic arch
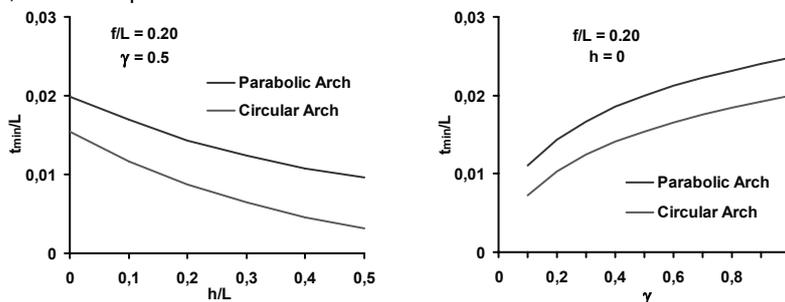
t_{min}/L decreases when h/L gets higher. When the fill above the crown become very high in comparison to self weigth and backfill, the load tends to a uniformly distributed load for which the parabolic arch is funicular

t_{min}/L gets higher when γ increases for h=0

COMPARISON BETWEEN PARABOLIC AND CIRCULAR ARCHES - 2
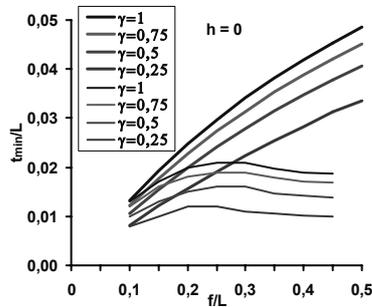
When f/L increases the two shapes show very different behaviour

For the parabolic arch the minimum thickness gets higher with f

The circular arch shows a more complex behaviour. The curves for different values of γ show that t_{min}/L increases with f/L up to a maximum point, then it gets lower. We can state that the sag ratio has no high influence on the minimum thickness except for low values of f/L. Besides, the horizontal part of the diagram, found in correspondence of (β_1, β_2), is missing due to the variation of L and therefore of t_{min}/L.

COMPARISON BETWEEN PARABOLIC AND CIRCULAR ARCHES - 3

In all the diagrams the curves relative to the circular arch give always values of t_{min} lower than those of the parabolic arch

We can conclude that the circular shape is more suitable to support dead loads (self weight and backfill)

CHECK UNDER DEAD LOADS - 1

The structural stability under dead loads can be evaluated first, by means of locating at least one thrust line which lies wholly within the arch profile.

As already said, it is possible to estimate the minimum thickness t_{min} the structure needs to contain such a thrust line.

If the actual depth t is greater than t_{min} the structure is safe, and its lifetime will depend on the decay of the materials.

The ratio between the actual depth and the theoretical minimum one can be assumed as a measure of safety when only dead loads are acting on the masonry arch; moreover the greater α, the greater the structure will be able to carry travelling loads

$$\alpha = \frac{t}{t_{min}}$$

CHECK UNDER DEAD LOADS - 2

The question is the safety margin with reference to a non proportional variation of the loads but also to a reduction of the effective cross-section, due to cracks or damages to the material.

In this latest case it is meaningful to evaluate the difference

$$\Delta = t - t_{min}$$

which represents the maximum crack or the maximum imperfection under dead loads

LIMIT ANALYSIS OF MASONRY ARCH BRIDGES

➤ **Limit analysis**

➤ **Funicular arch**

➤ **Check under dead loads**

➤ **Check under moving loads**

➤ **Check under horizontal actions**

TRAVELLING LOADS: Collapse Mechanism

Suppose that one thrust line in equilibrium with the dead loads exists which lies wholly within the masonry. Symmetry is not an issue here, because loads can lie anywhere on the structure

When the travelling loads are put in action and are increased from zero to the collapse value the line of thrust changes and at least four hinges form. At the point of collapse the thrust line must pass through the hinge points. At hinges at internal cross-sections the thrust line must be tangential to the arch profile

TRAVELLING LOADS: Iteration Procedure - 1

The mechanism of collapse and the corresponding load factor may be found in the usual way using an **iteration procedure**. This can be started giving a first mechanism and the corresponding diagram of velocity.

The equation of equilibrium can be written:

$$P_w + \lambda P_p = 0$$
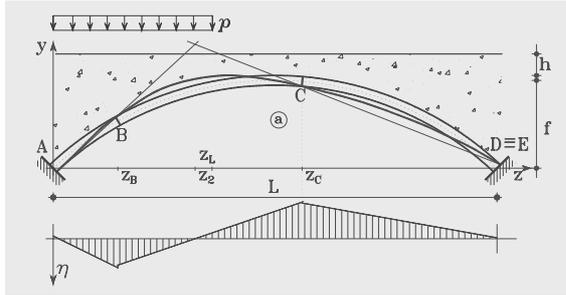
P_w and P_p are the **virtual powers** due to dead and travelling loads respectively

$$P_w = \int_0^L (w + w_1)\eta \; dz$$

$$P_p = \int_0^{z_p} p \; \eta \; dz$$

The **kinematic admissible** load factor can be deduced

$$\lambda = -P_w / P_p$$

TRAVELLING LOADS: Iteration Procedure - 2

Dead and travelling loads for each voussoirs

$$W_i = (w_i + w_{1i}) \cdot \Delta z_i$$

$$P_i = p_i \cdot \Delta z_i$$

Virtual powers:

$$P_w = \sum_1^n W_i \cdot \eta_i \qquad\qquad P_p = \sum_1^n P_i \cdot \eta_i$$

Kinematically admissible load factor:

$$\lambda = -\frac{\sum_1^n W_i \cdot \eta_i}{\sum_1^n P_i \cdot \eta_i}$$

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TRAVELLING LOADS: Iteration Procedure - 2
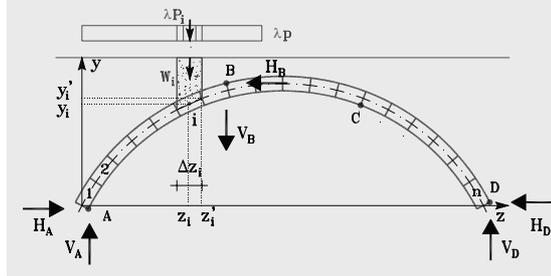
Loads: $W_i + \lambda P_i$

Equilibrium equations

$$H_A - H_D = 0$$

$$\sum_1^n W_i + \lambda \sum_1^n P_i - V_A - V_D = 0$$

$$\sum_1^n W_i \cdot (z_D - z_i) + \lambda \sum_1^n P_i(z_D - z_i) - V_A(z_D - z_A) - H_A(y_D - y_A) = 0$$

$$\sum_1^{i_B} W_j \cdot (z_B - z_j) + \lambda \sum_1^{i_B} P_j(z_B - z_j) - V_A(z_B - z_A) + H_A(y_B - y_A) = 0$$

We deduce V_A, V_D, H_A

TRAVELLING LOADS: Iteration Procedure - 3

Internal forces at section "i":

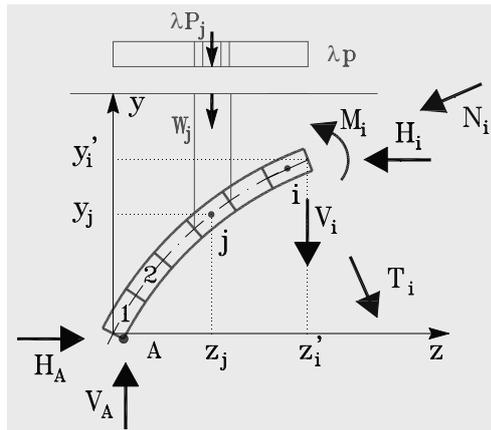$$V_i = V_A - \sum_1^i W_j - \lambda \sum_1^i P_j$$

$$H_i = H_A$$

Finally

$$T_i = V_i \cdot \cos\alpha_i - H_i \cdot \operatorname{sen}\alpha_i$$

$$N_i = V_i \cdot \operatorname{sen}\alpha_i + H_i \cdot \cos\alpha_i$$

$$M_i = V_A \cdot (z_i' - z_A) - H_A \cdot (y_i' - y_A) - \sum_1^i W_j(z_i' - z_j) - \lambda \sum_1^i P_j(z_i' - z_j)$$

We calculate the position of the resultant with reference to the centre line

$$e_i = M_i / N_i$$

TRAVELLING LOADS: Iteration Procedure - 2

Once λ is known we can evaluate:

1) the loads acting on the structures

2) the external reactions

3) the line of thrust passing through the assumed hinges

The value of λ is the collapse load factor if it is also statically admissible, i.e., if the line of thrust passing through the supposed hinges is elsewhere within the masonry
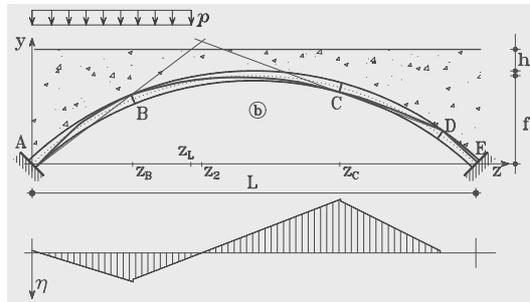
$$e_i \leq t/2$$

TRAVELLING LOADS: Iteration Procedure - 3
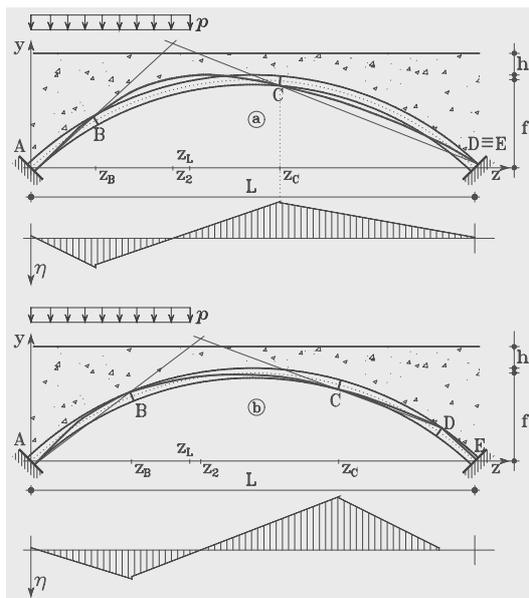
If λ is not a statically admissible factor, in the next step we move the hinges to the cross-sections where the distances between the line of thrust and the arch profile are maximum

The procedure will be repeated until the line of thrust is everywhere within the masonry

TRAVELLING LOADS: Numerical investigation

A uniform travelling load p acting from the center-point of the left springing to the section at z_p is considered, in order to simulate a load coming on to the bridge
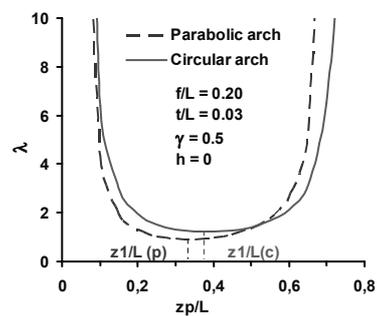
We assume

$$p = \frac{1}{L}\int_0^L (w + w_1)dz$$

With this kind of load the hinge A will certainly form at the left springing. Hinge D may or may not form at the right springing

Then, during each iteration we must move three hinges (B, C, and D)

TRAVELLING LOADS: λ_{min}

For each load condition the load factor λ and the corresponding collapse mechanism has been found

The diagram of λ versus z_p shows two characteristic points

z_1, which individualises the minimum value of λ

z_2, which individualises a vertical asymptote

TRAVELLING LOADS: z_1 and z_2

The diagrams show a flat central part around z_1, which leads to state that a load condition close to the worst is achieved for a wide range of load length

The diagrams also show a vertical asymptote at $z_p = z_2 \approx 0.70$. When z_p approaches L the stability of the arch increases.

If $z_p > z_2$ there is no danger of failure, because the hypothesized mechanism cannot be exploited under such a load condition. This is exactly true for the parabolic shape, whose centre line actually fits the funicular polygon of the travelling load.
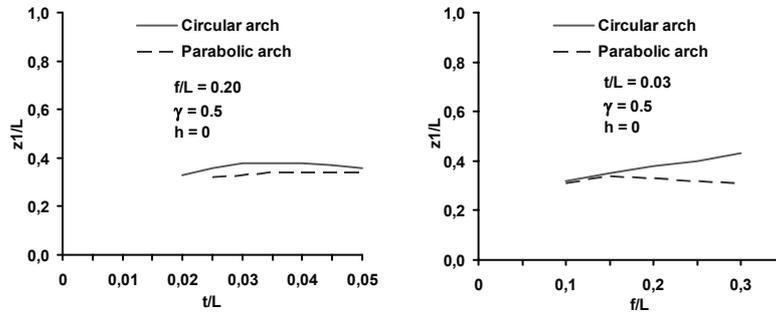
TRAVELLING LOADS: Hinge Locations
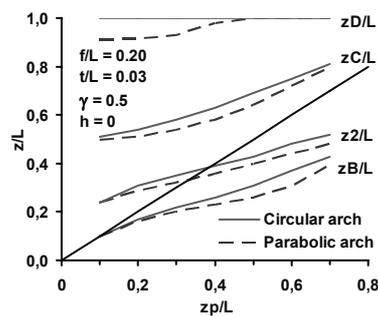
Hinge D may be not at the springing. This happens for the parabolic arch when $z_p < L/2$ and therefore also when $\lambda = \lambda_{min}$

Obviously, when $\lambda = \lambda_{min}$ it is $z_2 = z_p$. Actually, for a given mechanism, this is the load condition for which P_p assumes its maximum value and λ the minimum one

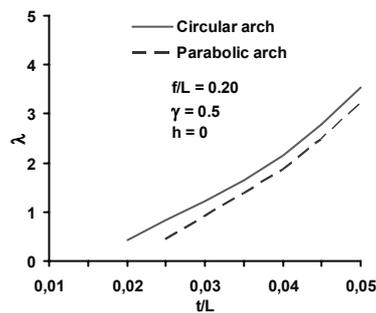# TRAVELLING LOADS: Influence of t and f on z_1

A wide numerical investigation on different values of the parameters t and f, showed that z_1 is almost constant and independent of t and f,, and it can be assumed equal to:

$z_1 = 0.33$ for the parabolic arch

$z_1 = 0.38$ for the circular arch

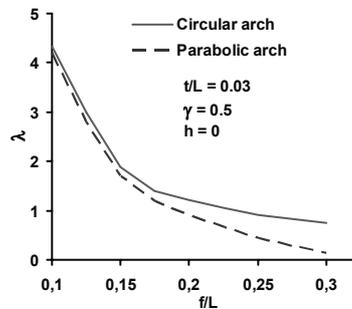# TRAVELLING LOADS: Influence of t on λ_{min}

The load factor λ gets higher with t

The relationship between λ and t is slightly different from a linear law

TRAVELLING LOADS: Influence of f on λ_{min}

The load factor λ increases quickly as f decreases

The reason for this behaviour is the following: when the arch has a small height, stresses due to dead loads are very high, so that the structure becomes less vulnerable to travelling loads. A very high load is necessary to move the line of thrust.

CHECK UNDER TRAVELLING LOADS

The check under dead loads has been carried out by comparing the effective thickness with the minimum one for which the line of thrust is wholly within the masonry

$$\alpha = \frac{t}{t_{min}}$$

It is interesting to find out what load factor λ corresponds to a given value of α

A numerical investigation has been carried out in order to evaluate the influence of α on the load factor λ for both parabolic and circular shape

TRAVELLING LOADS: λ versus α

λ increases with α. Both the shapes show a remarkable influence of γ on the value of λ: the load factor increases with γ
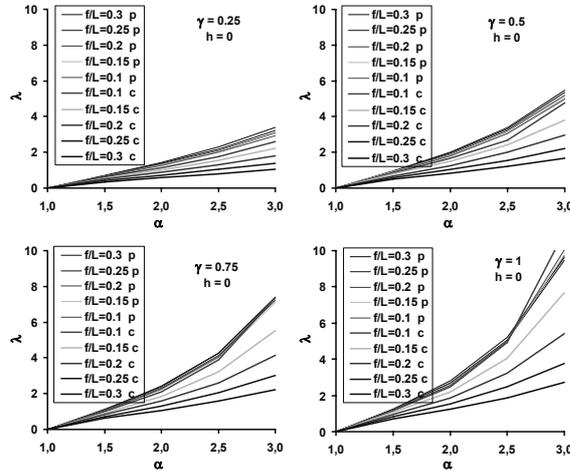
This influence is more important than we can see from the diagrams because of the assumed definition of the reference travelling load value, being equal to the mean dead load
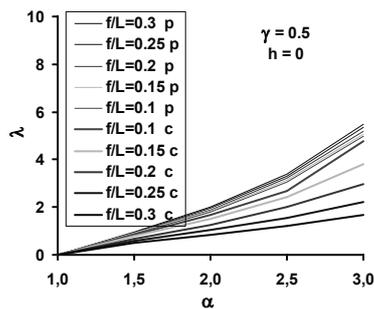
$$p = \frac{1}{L}\int_0^L (w + w_1)dz$$

TRAVELLING LOADS: Influence of f on λ-α

With reference to f the two shapes perform in different ways

The curves λ-α relative to the parabolic arch are quite independent of the height f (λ increases very little with f)

For the circular arch the influence of f on λ-α curves is not negligible. This time λ decreases with f and this influence increases with γ.

Finally, for a given α, the values of λ for the parabolic shape are always greater than the values of the circular shape

LIMIT ANALYSIS OF MASONRY ARCH BRIDGES

➢**Limit analysis**

➢**Funicular arch**

➢**Check under dead loads**

➢**Check under moving loads**

➢**Check under horizontal actions**


INTRODUCTION TO SEISMIC ANALYSIS - 1

The behaviour of the arch made up of rigid voussoir laid dry is very different from that of an elastic structure, especially when subject to seismic actions.

In fact, the vulnerability of an elastic structure under seismic loading is essentially related to the frequency content of the seismic input.

If the Fourier spectrum of this shows significant content at the main structural resonances, the structure is sensitive to it and may present high values of the amplification factor from the basement to the top.

Vice versa if the frequencies of the structure are not in the interval in which the seismic input spectrum shows important amplitudes, the earthquake will have not important effects on the structure

INTRODUCTION TO SEISMIC ANALYSIS - 2
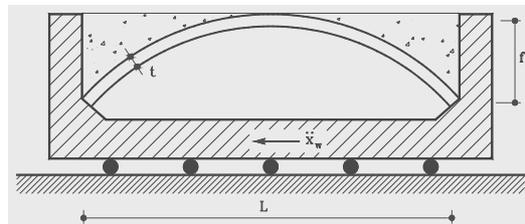
The study of an arch of rigid voussoir must be performed in a very different way. Because of its infinite rigidity, a stone arch does not show relative motion with respect to its base, until the amplitude of the external load is that to turn the structure into a mechanism.

From this consideration one can deduce that the safety check of stone arches can be studied as a static problem

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INTRODUCTION TO SEISMIC ANALYSIS - 2

If the arch is subject to the **self-weight only**, the horizontal forces are equal to the product between the mass of each voussoir and the horizontal acceleration and are applied at the centre gravity of each voussoir, i.e. at the same point of the vertical loads.

The structure is subjected to a **parallel forces system**, whose direction is determined by the ratio between the horizontal ground acceleration and the gravity acceleration.

It is as the springings of the arch were not at the same height, and the structure was fixed on a **inclined plane**

If the back-fill is acting on the structure the problem is quite complex. Suppose that its base is subjected to a **horizontal negative acceleration**

$$- \ddot{x}_g$$

Each voussoir is subjected to the inertial force due to its own mass, acting at the centre point of the voussoir. **Four models** are assumed to simulate the structure - back-fill interaction
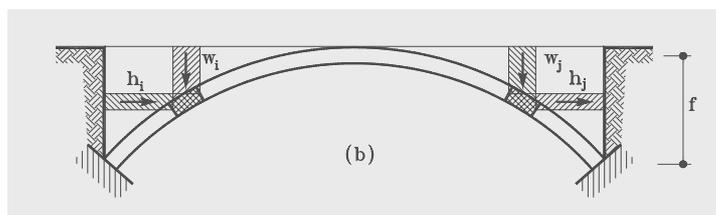
SEISMIC ACTION: Model M1

(a)
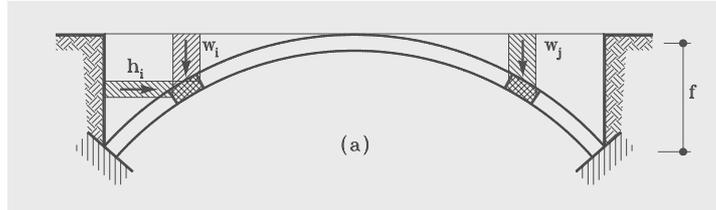
Model M1

each voussoir on the left is subjected to the inertial force due to an horizontal strip of the back-fill, and acting at its centre point. The length of this strip is assumed to be equal to the distance between the arch centre line and the vertical line passing through the left springing of the arch. The back-fill on the right tends to separate from the arch, because of its inertial forces. As a result the right half structure is subject to vertical loads and own inertial forces due to its own mass only

SEISMIC ACTION: Model M2

(b)

Model M2

the left half arch is loaded as in the model M1. The back-fill on the right is supposed to be attached to the structure. As a result the voussoir of the right half arch are subjected to the inertial forces due to an horizontal strip of the back-fill and acting at its centre point. The length of this strip is assumed to be equal to the distance between the centre line and the vertical line passing through the right springing
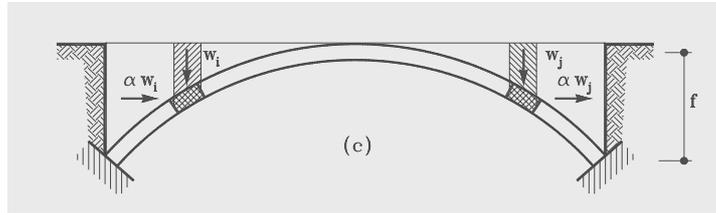
This load condition is anti-symmetric

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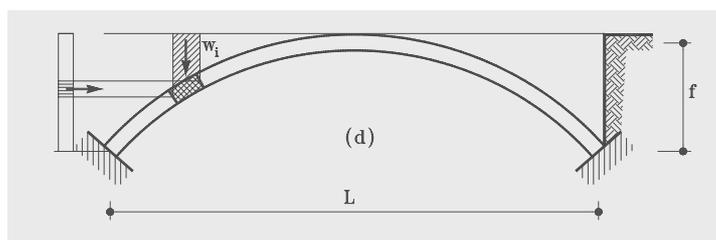
SEISMIC ACTION: Model M3

(c)

Model M3

each voussoir is subjected to an horizontal force equal to the vertical loads acting on it

Also this load condition is anti-symmetric
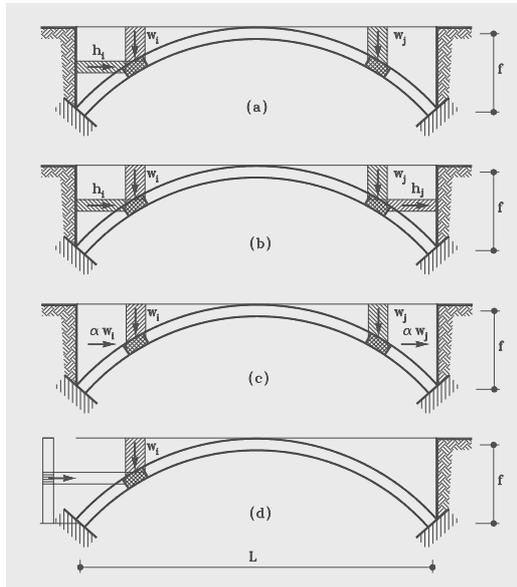
SEISMIC ACTION: Model M4

(d)

Model M4

the inertial load is simulated with a uniform load acting on the left half arch, whose resultant is equal to the total inertial load of structure - plus - back-fill

$$1/f \cdot \int_0^L q_v \cdot dz$$

SEISMIC ACTION MODELLING

The first model M1 seems to be the most realistic

Model M2 represents the opposite limit case with respect to M1

Models M3 and M4 are the most simple and useful for the preliminary check of the structure

In all the cases a reference value of the acceleration equal to the gravity acceleration has been assumed
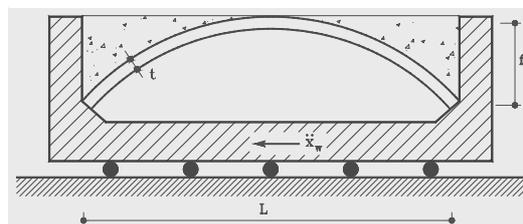
STRUCTURAL MODEL

Suppose that one thrust line in equilibrium with the dead loads exists which lies wholly within the masonry. The base of the arch is subjected to a negative horizontal acceleration .

In this condition the arch is subjected to the vertical loads and to the horizontal inertial forces due to the acceleration.

According to the safe theorem the structure is safe if a line of thrust in equilibrium with the external loads and lying wholly within the masonry can be found
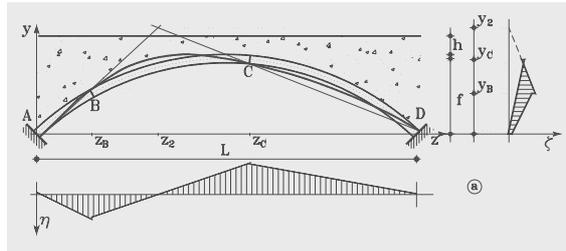
SEISMIC ACTION: Mechanism Model

In this case the horizontal loads play the role of the live loads, whereas the vertical ones are fixed.

The **uniqueness theorem** is still valid.

When the seismic loads are put in action and are increased from zero to the collapse value, the line of thrust changes and at least four hinges form. At the point of collapse the thrust line must pass through the hinge points. If the hinge is at an internal section the line of thrust must be tangential to the arch profile
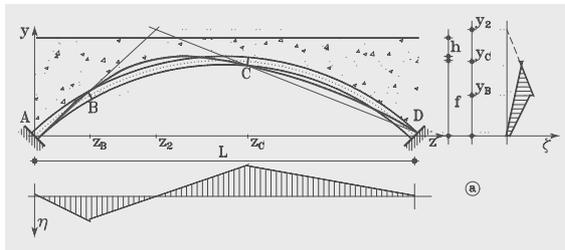
SEISMIC ACTION: Iteration Procedure - 1

Collapse mechanism and load factor can be found by using the same iteration procedure shown in the case of the vertical travelling load. This can be started giving a first mechanism and the corresponding diagrams of the velocity components. The equation of equilibrium can be written

$$P_v + \lambda\, P_h = 0$$

P_v and P_h are the virtual powers due to vertical and horizontal loads respectively

$$P_v = \int_0^L p_v \cdot \eta \; dz$$

$$P_h = \int_0^{z_L} p_h \cdot \zeta \; dz$$

$p_v(z)$ = the vertical dead load
$p_h(z)$ = the horizontal load,
$\eta(z)$ = vertical component of the velocity
$\varsigma(z)$ = the horizontal components of the velocity

SEISMIC ACTION: Iteration Procedure - 2

$$P_v + \lambda P_h = 0$$

The vertical loads have a stabilizing effect. In fact, because there are no terms with the internal forces acting at the hinges, which are still unknown, the equilibrium on the point of collapse is guaranteed by P_v which must be negative
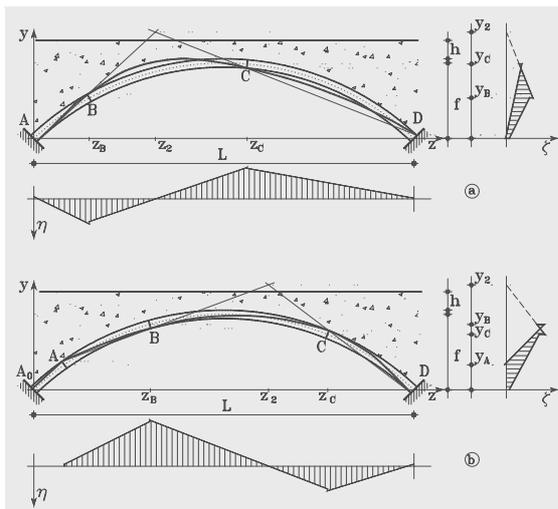
The **kinematically admissible** load factor can be deduced

$$\lambda = -P_v/P_h$$

and so the load intensity associated with the assumed mechanism.

The load being known, it is possible to find out the reactions and then the line of thrust passing through the assumed hinges.

SEISMIC ACTION: Iteration Procedure - 3

The value of λ is the collapse load factor only if it is also statically admissible, i.e., if the line of thrust, passing through the hypothesized hinges, is elsewhere within the masonry

If it is not, in the next step we must move the hinges to the sections where there are the maximum distances between the line of thrust and the arch profile

The load factors λ in the numerical investigation represent the ratio between the acceleration needed to turn the structure into a mechanism and the gravity acceleration g

SEISMIC ACTION: Hinge Locations
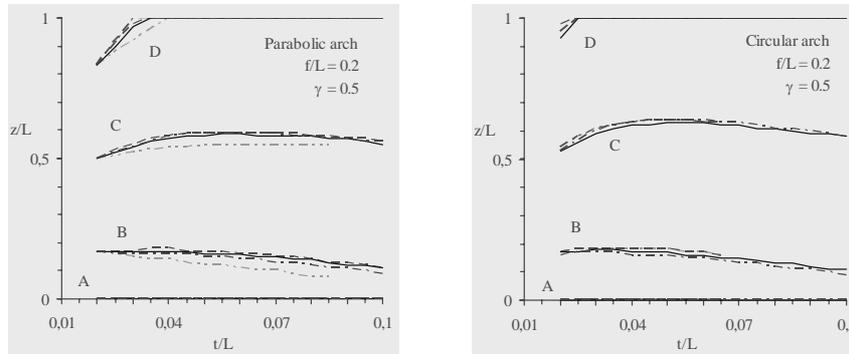
Usually **hinge A** forms always at the left springing, **hinge D** may not form at the right end

This is not always true, but depends on the values of γ

In the circular arch, for γ very low and high value of f/L, hinge A does not form at the left end, and hinge D is at the right springing
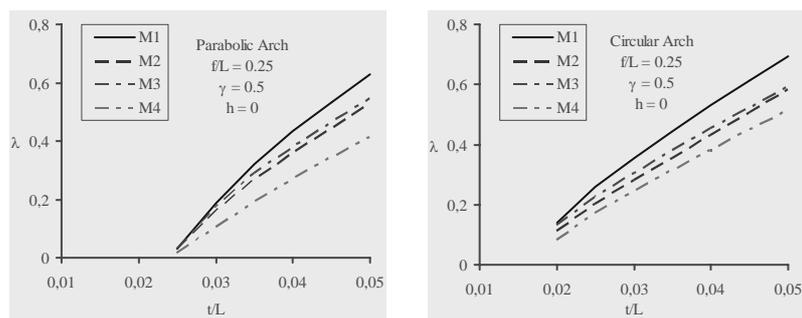
The influence of the seismic action modeling on the hinge locations is very low

SEISMIC ACTION: Influence of t on λ_{min}

As we expected, λ increases with t/L

The curves start from the minimum value of the depth under vertical dead loads

SEISMIC ACTION: Influence of f on λ_{min}
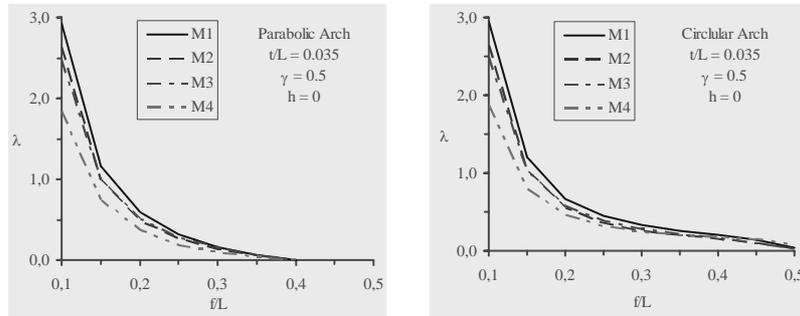
The load factor λ increases quickly as f/L decreases: when the arch has a small height, stresses due to dead loads are very high, and the structure becomes less vulnerable to live loads. A very high load is necessary to move the line of thrust

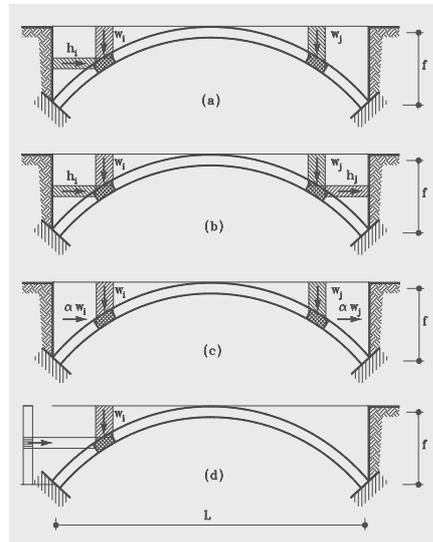The maximum value of f/L is related by the assumed value of t/L

SEISMIC ACTION: Influence of γ on λ_{min}
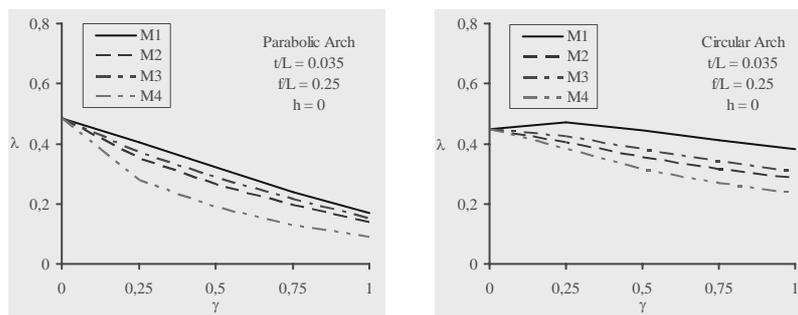
The relationship between λ and γ is slightly different from a linear law

When γ=0 the curves all start from the same value of λ, which represent the load factor for the arch subject to its self-weight only

SEISMIC ACTION : Comparison

The curves relative to model M1, which is to be considered as the better one, show values of λ always greater than the others.

The minimum values of λ are obtained by using the model M4, which represents the most conservative model of the seismic actions.

The simplest model, M3, for which the horizontal forces are proportional to the vertical ones for each voussoir, shows values of λ intermediate between those of cases M1 and M2

SEISMIC CHECK

The check of stone arches subject to seismic action loads is a static matter

So the analysis can be performed by using the same iteration procedure used fort he static loads

The most suitable loading pattern for seismic actions seams to be M1, but results very close to it are obtained also in case M3.

The load factors found in the numerical investigation represent the ratio between the acceleration needed to turn the structure into a mechanism and the gravity acceleration

So the safety check of a stone arch can be performed just by comparing the load factor with the acceleration peak of the actual earthquakes