Elastic bilayered plates with mismatch strain

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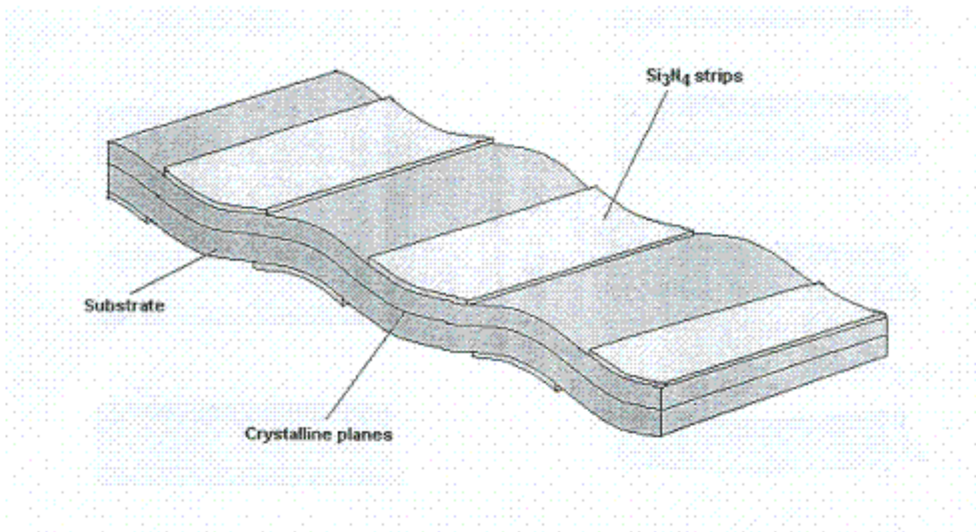
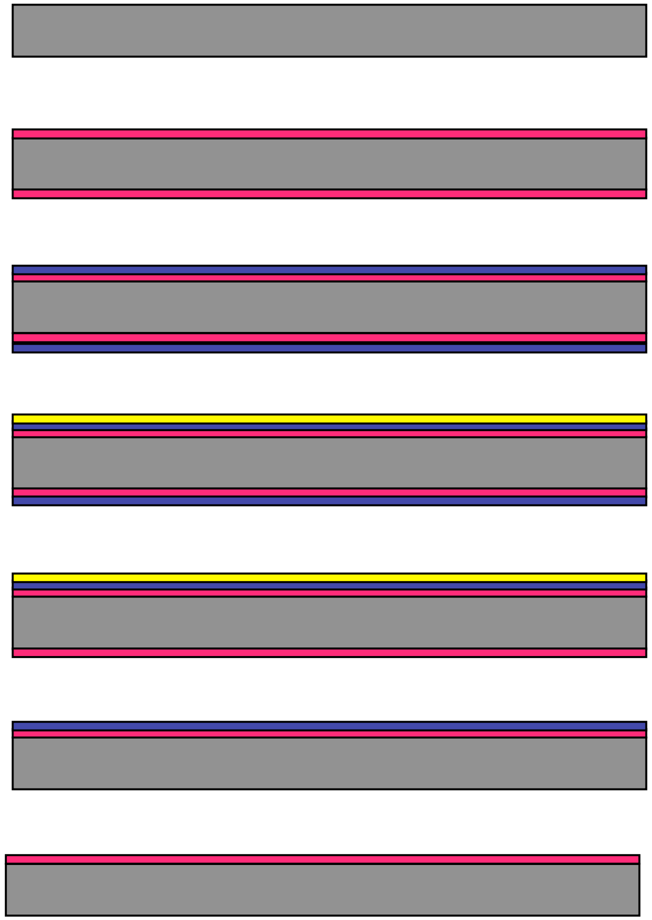


Surface micromachining



(111) oriented silicon monocrystal

Si_3N_4 deposition by CVD

Si_3O_2 deposition by CVD

Deposition of photoresistant polymer

Selective etching of Si_3O_2

Removal of Si_3N_4 and photoresistant polymer

Removal of Si_3O_2

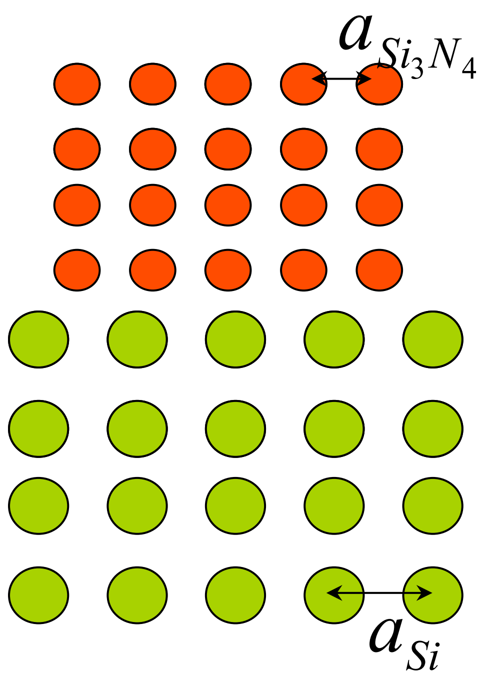
Deformation

$$\varepsilon_0 = \varepsilon_{INT} + \varepsilon_{EXT}$$

- Temperature induced deformation:

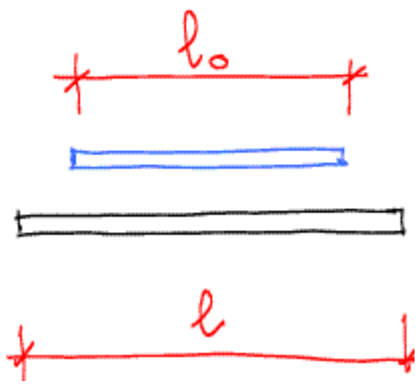
$$\varepsilon_{EXT} = \left(\alpha_{Si} - \alpha_{Si_3N_4}\right)\Delta T$$

- Deformation due to lattice mismatch :

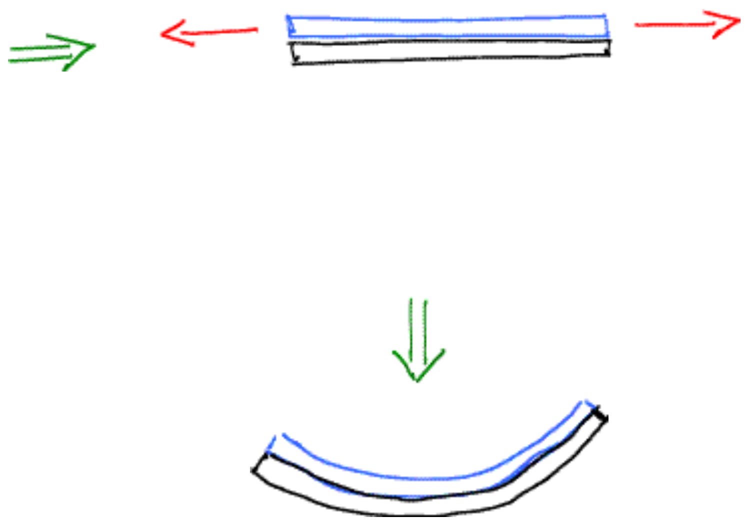


$$\varepsilon_{INT} = \frac{\left(a_{Si} - a_{Si_3N_4}\right)}{a_{Si_3N_4}}$$

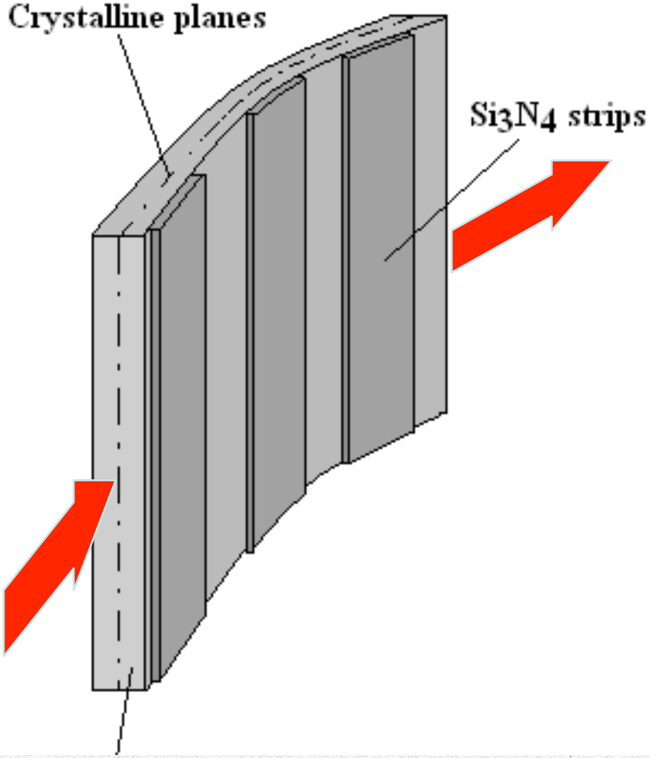
Mismatch deformation

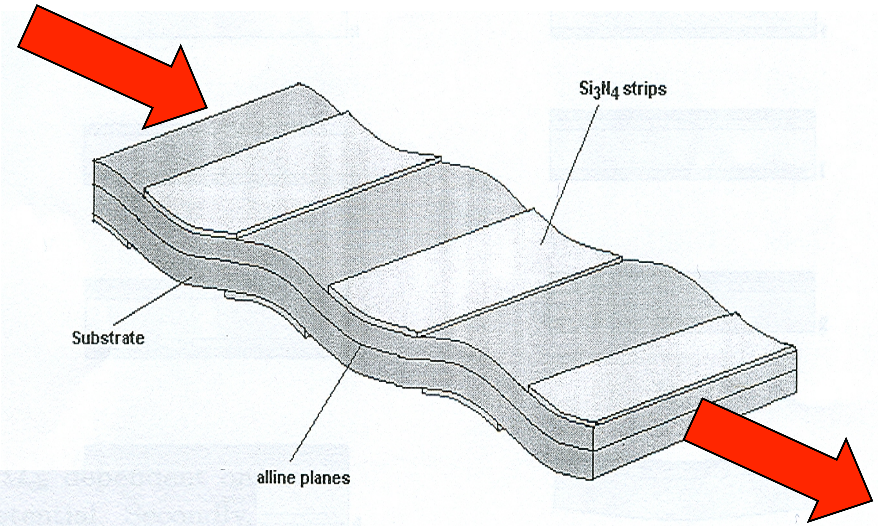


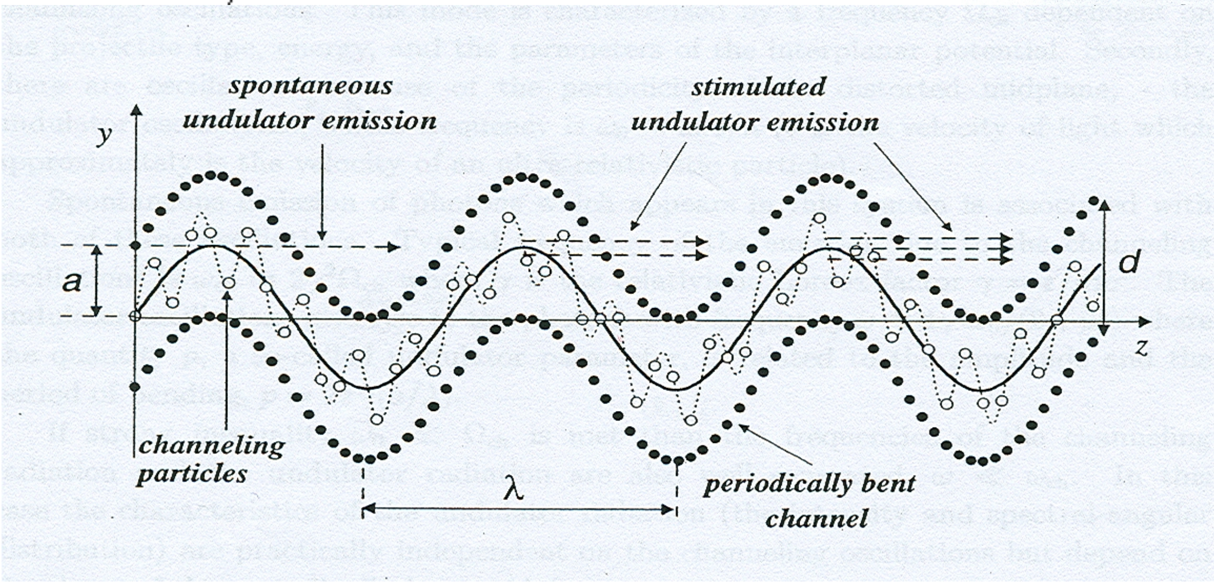
$$\varepsilon_0 = \frac{l - l_0}{l_0}$$



Applications

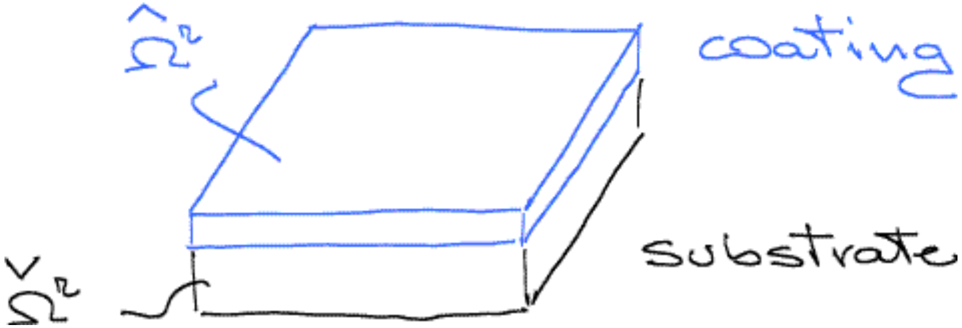






Undulator

3D-Mechanical modeling



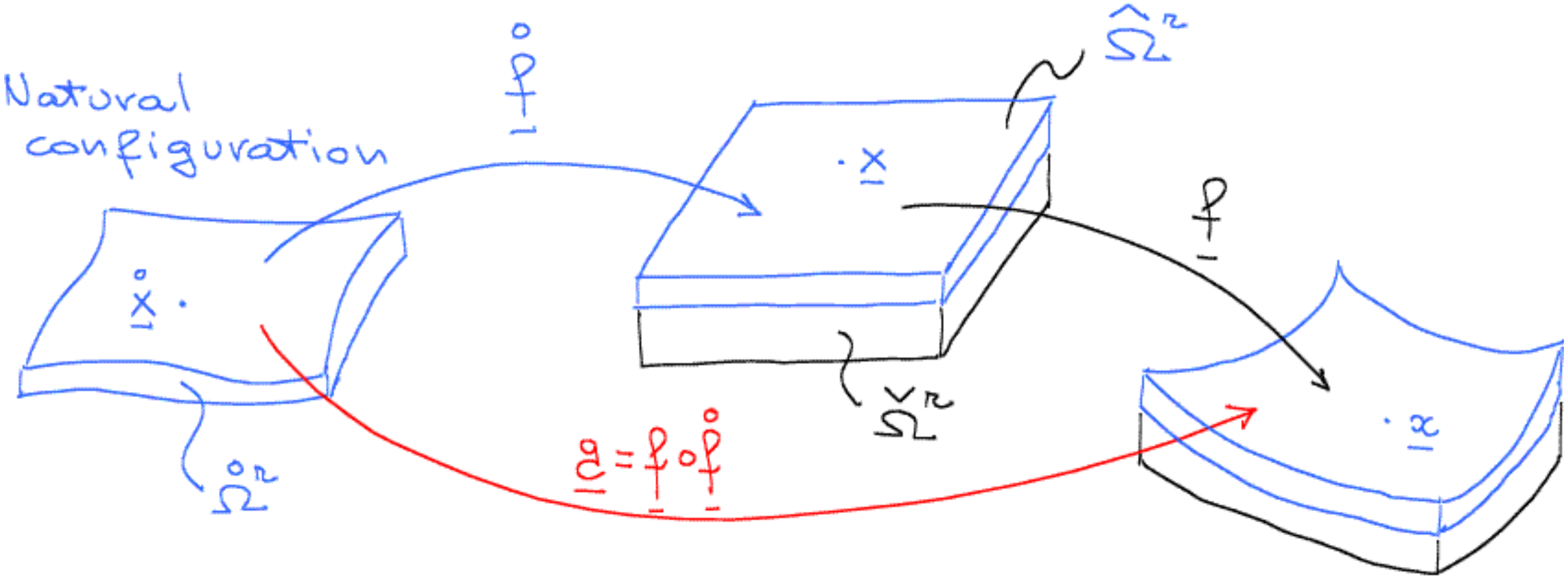
- The substrate is a linear, homogeneous, hyperelastic body.

  The reference configuration $\check{\Omega}^{2}$ is natural.

$$\check{W}(E) = \frac{1}{2}\, \mathbb{C}^{2} E \cdot E$$

- The coating is a linear, homogeneous, hyperelastic body.

  The reference configuration $\hat{\Omega}^{2}$ is NOT natural.

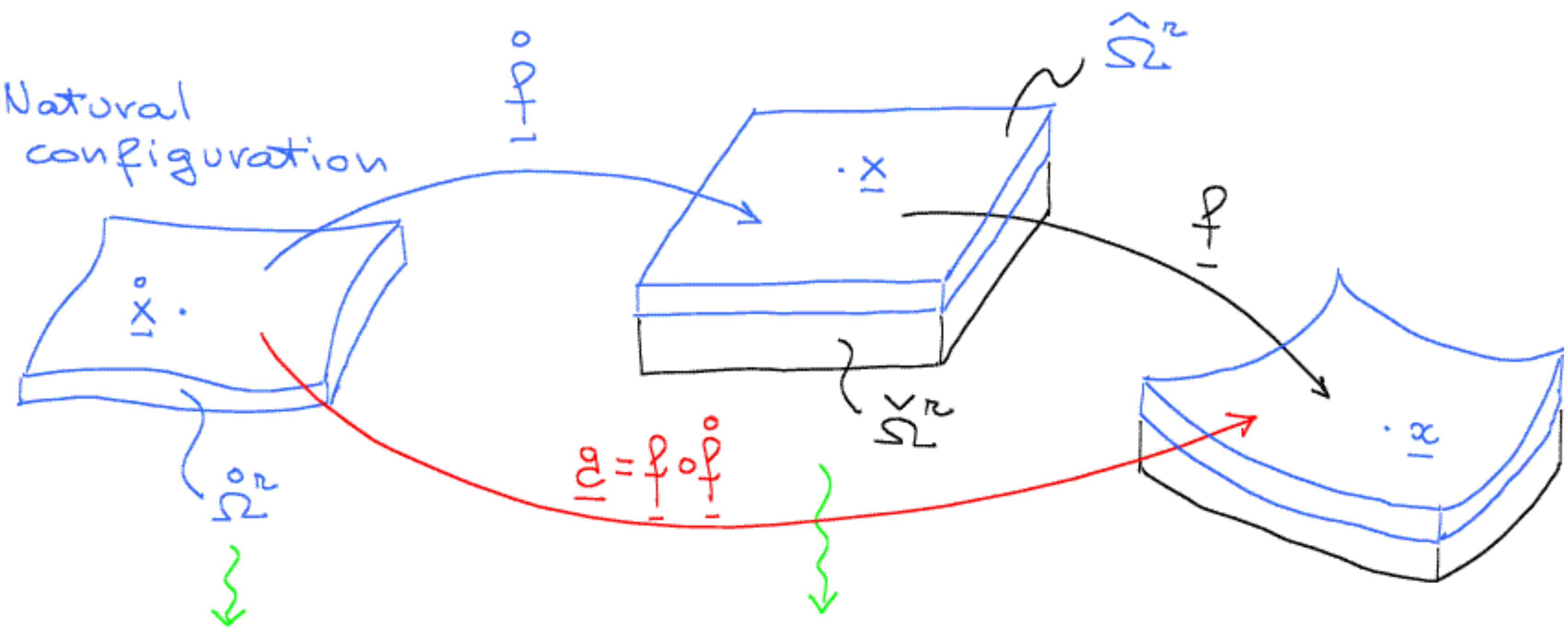


$$\Omega^r := \hat{\Omega}^r \cup \check{\Omega}^r$$

$$\mathring{\underline{f}} : \mathring{\Omega}^r \longrightarrow \hat{\Omega}^r \qquad \mathring{\underline{f}}(\mathring{\Omega}^r) = \hat{\Omega}^r$$

$$\underline{f} : \Omega^r \longrightarrow \mathbb{R}^3$$

$$\mathring{\underline{F}} = \nabla \mathring{\underline{f}} \qquad \underline{F} = \nabla \underline{f} \qquad \underline{G} = \nabla \underline{g} = F \mathring{F}$$



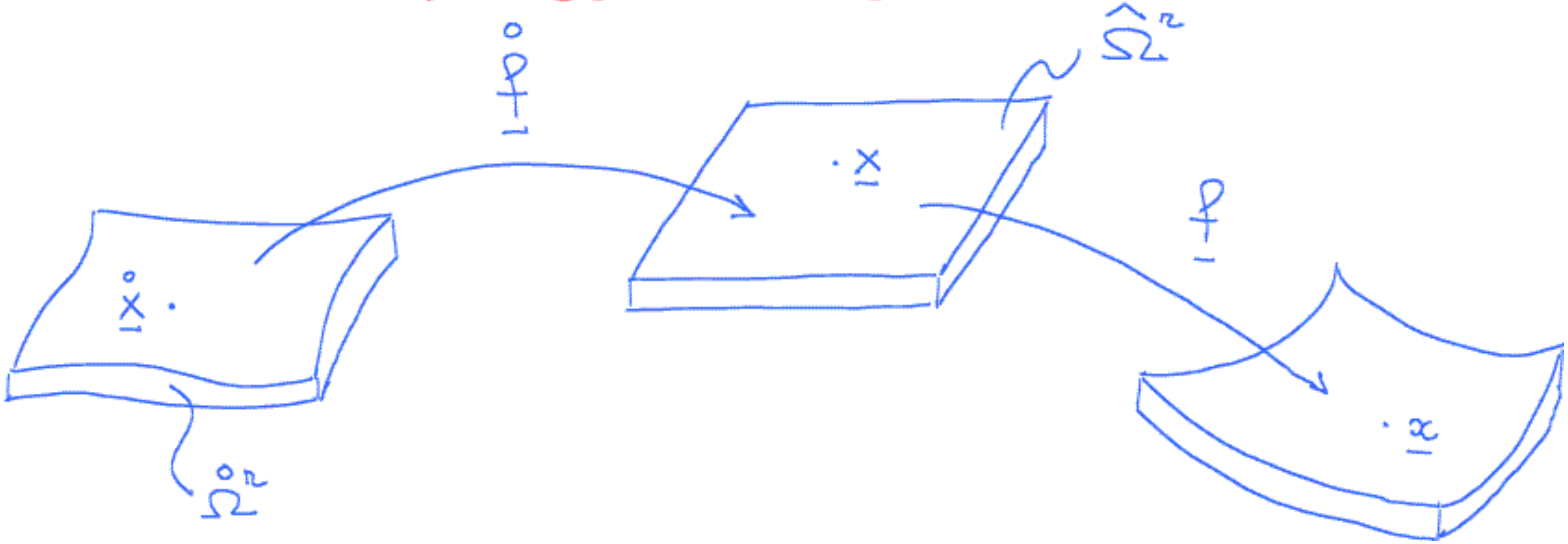
$$\int_{\mathring{\Omega}^r} \mathring{W}(\underline{G}(\mathring{\underline{x}}))\, d\mathring{x} \qquad\qquad \int_{\hat{\Omega}^r} \hat{W}(x, F(x))\, dx$$

$$\hat{W}(x, \underline{F}) := \det \mathring{\underline{F}}^{-1}(x)\, \mathring{W}(\underline{F}\, \mathring{\underline{F}}(\mathring{\underline{f}}^{-1}(\underline{x})))$$

$$\hat{\underline{S}} = D\hat{W}$$

$$\mathring{\underline{T}}^r(\underline{x}) = \hat{\underline{S}}(\underline{x}, \underline{I}) = D\mathring{W}(\mathring{\underline{F}})\, \mathring{\underline{F}}^T \det \mathring{\underline{F}}^{-1}$$

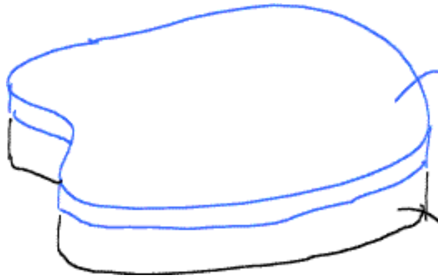
Linearization



$$\underline{F} = \nabla \underline{f} \quad \Rightarrow \quad \underline{F} = \underline{I} + \underline{H} \qquad \underline{E} = \tfrac{1}{2}\left(\underline{H} + \underline{H}^T\right)$$

$$\hat{W}(\underline{x}, \underline{F}) = \hat{W}(\underline{x}, \underline{I}) + \underbrace{D\hat{W}(\underline{x}, \underline{I})}_{\mathring{\underline{T}}^r} \cdot \underline{H} + \underbrace{\tfrac{1}{2} D^2\hat{W}(\underline{x}, \underline{I})\, \underline{H} \cdot \underline{H}}_{\frac{1}{2} \underline{H}\,\mathring{\underline{T}}^r \cdot \underline{H} + \frac{1}{2} \mathbb{L}^r \underline{E} \cdot \underline{E}} + o\left(|\underline{H}|^2\right)$$

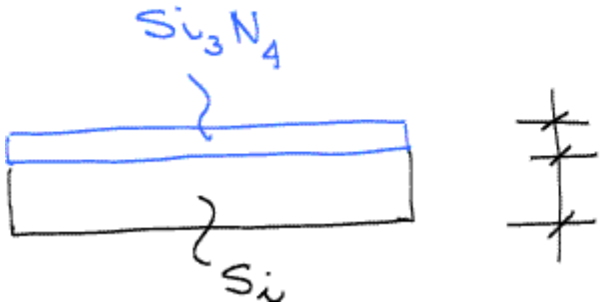
$$\hat{W}(x, \underline{F}) = \hat{W}(\underline{x}, \underline{I}) + \mathring{\underline{T}}^r \cdot \underline{H} + \tfrac{1}{2}\, \underline{H}\,\mathring{\underline{T}}^r \cdot \underline{H} + \tfrac{1}{2}\, \mathbb{L}^r \underline{E} \cdot \underline{E}$$



$$\hat{\Omega}^\varepsilon = \omega \times (0, \hat{h}^\varepsilon)$$

$$\check{\Omega}^\varepsilon = \omega \times (-\check{h}^\varepsilon, 0)$$

$$\mathcal{J}^\varepsilon(u) = \int_{\hat{\Omega}^\varepsilon} \mathring{T}^\varepsilon \cdot Eu + \frac{1}{2} \nabla u \mathring{T}^\varepsilon \cdot \nabla u + \frac{1}{2} \mathbb{L}^\varepsilon Eu \cdot Eu \, dx \; + \int_{\check{\Omega}^\varepsilon} \frac{1}{2} \mathbb{C}^\varepsilon Eu \cdot Eu \, dx$$



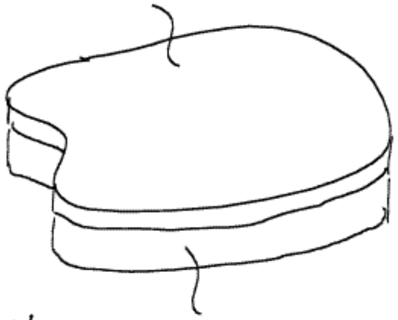
$$\hat{h}^\varepsilon = 100 \div 300 \text{ nm}$$

$$\check{h}^\varepsilon = 200 \div 300 \ \mu\text{m}$$

$$\varepsilon^\varepsilon := \frac{\check{h}^\varepsilon}{\operatorname{diam} \omega} \approx 10^{-2} \div 10^{-3}$$

$$\frac{\hat{h}^\varepsilon}{\check{h}^\varepsilon} \approx 10^{-3} \; !$$

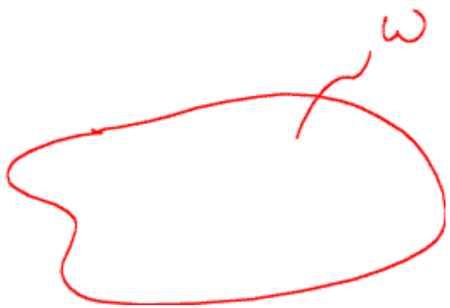
$$\hat{\Omega}^\varepsilon = \omega \times (0, \hat{h}^\varepsilon)$$



$$\check{\Omega}^\varepsilon = \omega \times (-\check{h}^\varepsilon, 0)$$

$$\min_{u \in \mathcal{A}^\varepsilon} \mathcal{J}^\varepsilon(u) \qquad (P^\varepsilon)$$

$$\mathcal{J}^\varepsilon(u^\varepsilon) = \min_{u \in \mathcal{A}^\varepsilon} \mathcal{J}^\varepsilon(u)$$



$$\min_{u \in \mathcal{A}^0} \mathcal{J}^0(u) \qquad (P^0)$$

$$\mathcal{J}^0(u^0) = \min_{u \in \mathcal{A}^0} \mathcal{J}^0(u)$$

How to choose (P^0) ?

(P^0) s.t. $u^\varepsilon \approx u^0$

Plate theory via variational convergence

$$P_\varepsilon \xrightarrow{v.c.} P_0$$

u_ε solution of P_ε $\qquad \varepsilon \in (0,1]$

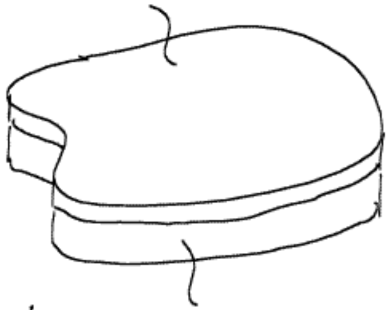
u_0 solution of P_0

if u_ε converges $\Rightarrow$ $u_\varepsilon \longrightarrow u_0$

How to use variational convergence?

$$P_\varepsilon \xrightarrow{\text{v.c.}} P_0 \Rightarrow u_\varepsilon \longrightarrow u_0$$

$$\hat{\Omega}^r = \omega \times (0, \hat{h}^r)$$



$$\check{\Omega}^r = \omega \times (-\check{h}^r, 0)$$

$$\varepsilon^r := \frac{\check{h}^r}{\operatorname{diam} \omega} \approx 10^{-2} \div 10^{-3}$$

$$\mathcal{J}^r(u^r) = \min_{u \in \mathcal{A}^r} \mathcal{J}^r(u) \qquad (P^r)$$

Choose P_ε so that:

1) P_ε variationally converges

2) u_ε = solution of P_ε converges

3) $P_{\varepsilon^r} \equiv P^r$

P., Podio-Guidugli
in preparation

$$P_\varepsilon \xrightarrow{\text{v.c.}} P_0$$

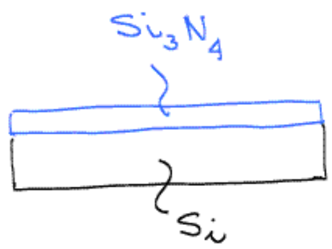
$$u_\varepsilon \longrightarrow u_0$$

$\Rightarrow$ $u_0 \approx u_\varepsilon$ for ε small

$$u_0 \approx u_{\varepsilon^r} = u^r$$

Definition of P_ε $\begin{cases} 1) \text{ Def. } \Omega_\varepsilon \\ 2) \text{ Def. } \mathcal{J}_\varepsilon \end{cases}$

$$\varepsilon^r := \frac{\check{h}^r}{\operatorname{diam}\omega} \approx 10^{-2} \div 10^{-3}$$



$\hat{h}^r = 100 \div 300$ nm

$\check{h}^r = 200 \div 300$ μm

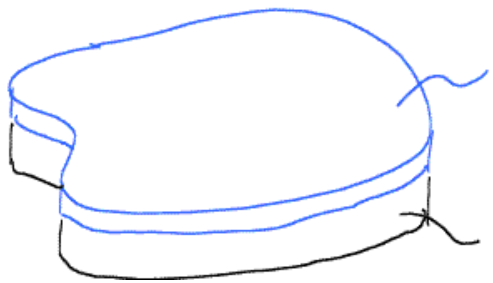
$$\frac{\hat{h}^r}{\check{h}^r} \approx 10^{-3}$$

$$\check{h}^r = \operatorname{diam}\omega \; \varepsilon^r \implies \check{h}^\varepsilon := \operatorname{diam}\omega \; \varepsilon$$

$$\hat{h}^r = \operatorname{diam}\omega \; (\varepsilon^r)^{2s+1} \implies \hat{h}^\varepsilon = \operatorname{diam}\omega \; \varepsilon^{2s+1}$$

$s \approx 1 \div 3/2$

Ω_ε



$\hat{\Omega}^\varepsilon = \omega \times (0, \hat{h}\,\varepsilon^{2s+1})$ $\quad s > 0$

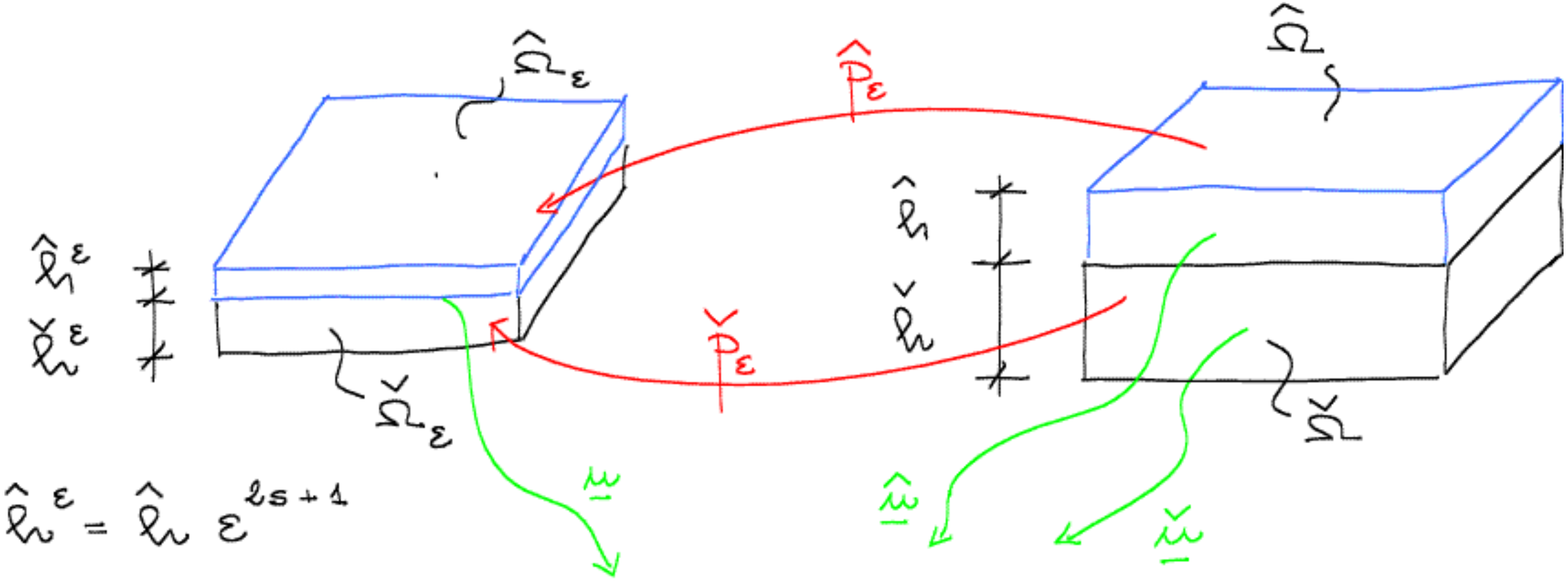
$\check{\Omega}^\varepsilon = \omega \times (-\check{h}\,\varepsilon, 0)$

$$\hat{h} = \check{h} = \mathrm{diam}\ \omega$$

Definition of P_ε $\begin{cases} 1) \text{ Def. } \Omega_\varepsilon \ \checkmark \\ 2) \text{ Def. } \mathcal{F}_\varepsilon \end{cases}$

$$P_\varepsilon \xrightarrow{\text{v.c.}} P_0 \Rightarrow u_\varepsilon \longrightarrow u_0$$

Change of variables



$$\hat{h}^\varepsilon = \hat{h}\,\varepsilon^{2s+1}$$

$$\check{h}^\varepsilon = \check{h}\,\varepsilon$$

$$\underline{\hat{u}}^\varepsilon = \underline{u} \circ \hat{p}_\varepsilon \qquad \underline{\check{u}}^\varepsilon = \underline{u} \circ \check{p}_\varepsilon$$

$$(\nabla u) \circ \hat{p}_\varepsilon = \nabla \hat{u}\; \hat{P}^{\varepsilon^{-1}} =: \hat{H}^\varepsilon \hat{u} \qquad \hat{P}^\varepsilon = \nabla \hat{p}^\varepsilon = \operatorname{diag}(1, 1, \varepsilon^{2s+1})$$

$$(\nabla u) \circ \check{p}_\varepsilon = \nabla \check{u}\; \check{P}^{\varepsilon^{-1}} =: \check{H}^\varepsilon \check{u} \qquad \check{P}^\varepsilon = \nabla \check{p}^\varepsilon = \operatorname{diag}(1, 1, \varepsilon)$$

$$\hat{H}^\varepsilon \hat{u} = \hat{E}^\varepsilon \hat{u} := \operatorname{sym} \hat{H}^\varepsilon \hat{u} \qquad \check{E}^\varepsilon \check{u} := \operatorname{sym} \left. \check{H}^\varepsilon \check{u} \;\; \frac{D_3 \underline{\check{u}}^\varepsilon}{\varepsilon} \right)$$

$$\mathcal{J}^{\eta}(u) = \int_{\hat{\Omega}^{\eta}} \overset{\circ}{T}^{\eta} \cdot Eu + \frac{1}{2} \nabla u \overset{\circ}{T}^{\eta} \cdot \nabla u + \frac{1}{2} \mathbb{L}^{\eta} Eu \cdot Eu \, dx + \int_{\check{\Omega}^{\eta}} \frac{1}{2} \mathbb{C}^{\eta} Eu \cdot Eu \, dx$$

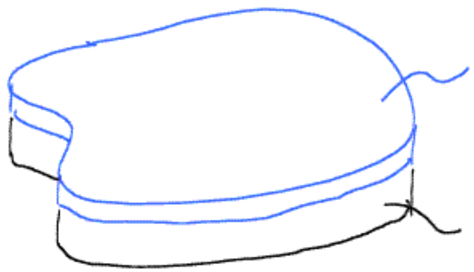
$$\hat{\Omega}^{\eta} = \hat{\Omega}^{\varepsilon^{\eta}} \qquad \check{\Omega}^{\eta} = \check{\Omega}^{\varepsilon^{\eta}}$$

$$= \varepsilon^{\eta^{2s+1}} \int_{\hat{\Omega}} \overset{\circ}{T}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \hat{H}^{\varepsilon^{\eta}} \hat{u} \overset{\circ}{T}^{\eta} \circ p^{\varepsilon^{\eta}} \cdot \hat{H}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \mathbb{L}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \hat{E}^{\varepsilon^{\eta}} \hat{u} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u} \, dx +$$

$$+ \varepsilon^{\eta} \int_{\check{\Omega}} \frac{1}{2} \mathbb{C}^{\eta} \circ \check{p}^{\varepsilon^{\eta}} \check{E}^{\varepsilon^{\eta}} \check{u} \cdot \check{E}^{\varepsilon^{\eta}} \check{u} \, dx$$

$$\mathcal{J}^{\eta}(u) = \varepsilon^{2s+1}_{\eta} \int_{\hat{\Omega}} \overset{\circ}{T}{}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \hat{H}^{\varepsilon^{\eta}} \hat{u}\, \overset{\circ}{T}{}^{\eta} \circ p^{\varepsilon^{\eta}} \cdot \hat{H}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \mathbb{L}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \hat{E}^{\varepsilon^{\eta}} \hat{u} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u}\, dx +$$

$$+ \varepsilon^{\eta} \int_{\check{\Omega}} \frac{1}{2} \mathbb{C}^{\eta} \circ \check{p}^{\varepsilon^{\eta}} \check{E}^{\varepsilon^{\eta}} \check{u} \cdot \check{E}^{\varepsilon^{\eta}} \check{u}\, dx$$



$\hat{\Omega}^{\varepsilon} = \omega \times (0, \hat{h}\, \varepsilon^{2s+1})$

$\mathcal{J}^{\varepsilon}(\hat{u}, \check{u}) :=$

$\check{\Omega}^{\varepsilon} = \omega \times (-\check{h}\, \varepsilon, 0)$

$\cdot\ \hat{H}^{\varepsilon} \hat{u}$

Definition of P_{ε} $\begin{cases} 1)\ \text{Def. } \Omega_{\varepsilon} \checkmark \\ 2)\ \text{Def. } \mathcal{J}_{\varepsilon} \end{cases}$ dx

$P_{\varepsilon^{\eta}} = P^{\eta}$

$\dot{\Omega}$ $\mathbb{C}$ x

$$\mathcal{J}^{\eta}(u) = \varepsilon^{\eta(2s+1)} \int_{\hat{\Omega}} \mathring{T}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \hat{H}^{\varepsilon^{\eta}} \hat{u}\, \mathring{T}^{\eta} \circ p^{\varepsilon^{\eta}} \cdot \hat{H}^{\varepsilon^{\eta}} \hat{u} + \frac{1}{2} \mathbb{L}^{\eta} \circ \hat{p}^{\varepsilon^{\eta}} \hat{E}^{\varepsilon^{\eta}} \hat{u} \cdot \hat{E}^{\varepsilon^{\eta}} \hat{u}\, dx +$$

$$+ \varepsilon^{\eta} \int_{\check{\Omega}} \frac{1}{2} \mathbb{C}^{\eta} \circ \check{p}^{\varepsilon^{\eta}} \check{E}^{\varepsilon^{\eta}} \check{u} \cdot \check{E}^{\varepsilon^{\eta}} \check{u}\, dx$$

$$\mathcal{J}^{\varepsilon}(\hat{u}, \check{u}) = \varepsilon^{2s} \int_{\hat{\Omega}} \varepsilon^{t} \mathring{T} \cdot \hat{E}^{\varepsilon} \hat{u} + \varepsilon^{q+t} K \hat{H}^{\varepsilon} \hat{u}\, \mathring{T} \cdot \hat{H}^{\varepsilon} \hat{u} + \frac{1}{2} \frac{1}{\varepsilon^{2\eta}} \mathbb{L} \hat{E}^{\varepsilon} \hat{u} \cdot \hat{E}^{\varepsilon} \hat{u}\, dx$$

$$+ \int_{\check{\Omega}} \frac{1}{2} \mathbb{C} \check{E}^{\varepsilon} \check{u} \cdot \check{E}^{\varepsilon} \check{u}\, dx$$

$$\mathcal{J}^{\eta}(u) = \varepsilon^{\eta} \mathcal{J}^{\varepsilon^{\eta}}(\hat{u}, \check{u})$$

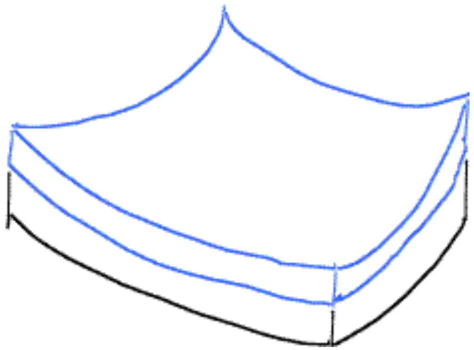
$$(P^{\eta}) = (P^{\varepsilon^{\eta}})$$

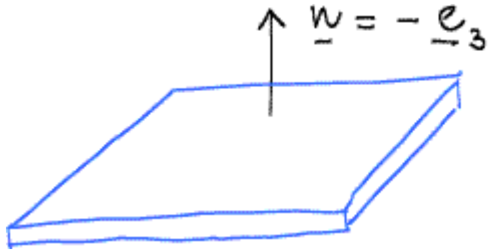
Definition of P_{ε} $\begin{cases} 1) \text{ Def. } \Omega_{\varepsilon} \ \checkmark \\ 2) \text{ Def. } \mathcal{J}^{\varepsilon} \ \checkmark \end{cases}$

$$\mathcal{J}^\varepsilon(\hat{u},\check{u}) = \varepsilon^{2s}\int_{\hat{\Omega}} \varepsilon^t \mathring{T}\cdot\hat{E}^\varepsilon\hat{u} + \varepsilon^{q+t} K\,\hat{H}^\varepsilon\hat{u}\,\mathring{T}\cdot\hat{H}^\varepsilon\hat{u} + \frac{1}{2}\frac{1}{\varepsilon^2\eta}\mathbb{L}\,\hat{E}^\varepsilon\hat{u}\cdot\hat{E}^\varepsilon\hat{u}\,dx$$

$$+\int_{\check{\Omega}}\frac{1}{2}\,\mathbb{C}\,\check{E}^\varepsilon\check{u}\cdot\check{E}^\varepsilon\check{u}\,dx$$

Assumptions on $\mathring{T}$





$$\mathring{T}_{i3} = 0$$

$$\mathring{T}_{\alpha\beta}(x)\,a_\alpha a_\beta \geq \tau_{min}\,a_\alpha a_\alpha \qquad \forall a_1, a_2 \in \mathbb{R}$$

with $\tau_{min} > 0$

Assumptions on the elasticity tensors

$\exists\ \hat{c}_L > 0$ s.t. $\mathbb{L}\,E\cdot E \geq \hat{c}_L |E|^2 \quad \forall E \in \mathbb{R}^{3\times3}_{sym}$

$\exists\ \check{c} > 0$ s.t. $\mathbb{C}\,E\cdot E \geq \check{c}\,|E|^2 \quad \forall E \in \mathbb{R}^{3\times3}_{sym}$

P_ε so that:

1) P_ε variationally converges

2) u_ε = solution of P_ε converges ✓

3) $P_{\varepsilon^2} \equiv P^2$ ✓

if $t + \eta + s \geq 0 \rightarrow$ 2)

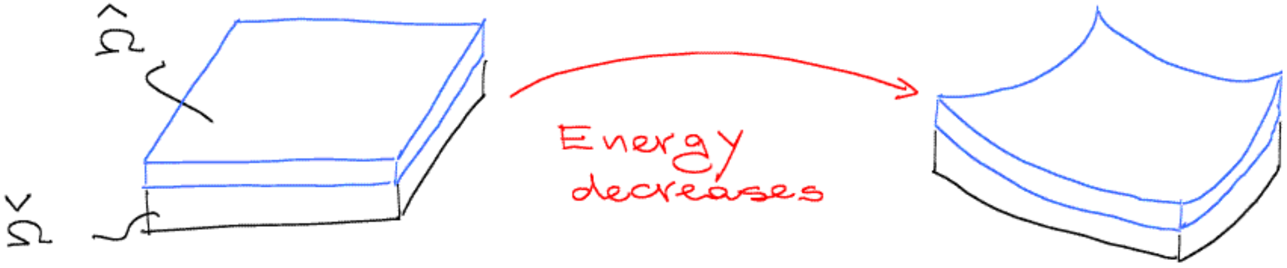
$$\mathcal{A} := \{(\hat{u}, \check{u}) \in H^1_D(\hat{\Omega}) \times H^1_D(\check{\Omega}) : \hat{u}(\cdot, \cdot, 0) = \check{u}(\cdot, \cdot, 0)\}$$

if $t + \eta + s \geq 0$

$$(\hat{u}^\varepsilon, \check{u}^\varepsilon) \in \mathcal{A} \quad \text{s.t.} \quad \sup_\varepsilon \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) < +\infty$$

$$\rightarrow \quad \sup_\varepsilon \int_{\hat{\Omega}} \left| \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \right|^2 dx + \int_{\check{\Omega}} |\check{E}^\varepsilon \check{u}^\varepsilon|^2 dx < +\infty$$

Importance of the linear term



$$\mathcal{J}^\varepsilon(\hat u,\check u) = \varepsilon^{2s}\int_{\hat\Omega} \varepsilon^{t}\,\mathring{T}\cdot \hat E^\varepsilon \hat u + \varepsilon^{q+t} K\, \overbrace{\hat H^\varepsilon \hat u\, \mathring{T}\cdot \hat H^\varepsilon \hat u}^{\geq 0} + \frac{1}{2}\,\frac{1}{\varepsilon^{2q}}\, \overbrace{\mathbb{L}\,\hat E^\varepsilon \hat u\cdot \hat E^\varepsilon \hat u}^{\geq 0}\, dx$$

$$+ \int_{\check\Omega} \frac{1}{2}\, \underbrace{\mathbb{C}\,\check E^\varepsilon \check u\cdot \check E^\varepsilon \check u}_{\geq 0}\, dx$$

$$H\mathring{T}\cdot H \geq \sum_i \sum_\alpha \mathring{\tau}_{min} |H_{i\alpha}|^2 \geq 0$$

$$\mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) = \varepsilon^{2s} \int_{\hat{\Omega}} \varepsilon^t \mathring{T} \cdot \hat{E}^\varepsilon \hat{u}^\varepsilon + \dots$$

$$= \varepsilon^{t+\eta+s} \int_{\hat{\Omega}} \mathring{T} \cdot \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta-s}} + \dots$$

if $\quad t+\eta+s \geq 0$

$(\hat{u}^\varepsilon, \check{u}^\varepsilon) \in \mathcal{A}$ s.t. $\sup_\varepsilon \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) < +\infty$

$$\Rightarrow \quad \sup_\varepsilon \int_{\hat{\Omega}} \left| \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta-s}} \right|^2 dx + \int_{\check{\Omega}} |\check{E}^\varepsilon \check{u}^\varepsilon|^2 dx < +\infty$$

$$t+\eta+s = 0$$

$$\mathcal{J}^\varepsilon(\hat{u},\check{u}) = \int_{\hat{\Omega}} \mathring{T} \cdot \frac{\hat{E}^\varepsilon \hat{u}}{\varepsilon^{\eta-s}} + k \frac{\hat{H}^\varepsilon \hat{u}}{\varepsilon^{(\eta-s-q)/2}} \mathring{T} \cdot \frac{\hat{H}^\varepsilon \hat{u}}{\varepsilon^{(\eta-s-q)/2}} + \frac{1}{2} \mathbb{L} \frac{\hat{E}^\varepsilon \hat{u}}{\varepsilon^{\eta-s}} \cdot \frac{\hat{E}^\varepsilon \hat{u}}{\varepsilon^{\eta-s}} \, dx$$

$$+ \int_{\check{\Omega}} \frac{1}{2} \mathbb{C} \check{E}^\varepsilon \check{u} \cdot \check{E}^\varepsilon \check{u} \, dx$$

Rigidity $\quad \sup_\varepsilon \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) < +\infty$

- "Material" $\quad \sup_\varepsilon \int_{\hat{\Omega}} \left| \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta-s}} \right|^2 dx + \int_{\check{\Omega}} |\check{E}^\varepsilon \check{u}^\varepsilon|^2 dx < +\infty$

- "Retraction" $\quad \sup_\varepsilon \int_{\hat{\Omega}} \left| \frac{(\hat{H}^\varepsilon \hat{u}^\varepsilon)_{i\alpha}}{\varepsilon^{(\eta-s-q)/2}} \right|^2 dx < +\infty \Rightarrow$

"Material Rigidity"

$$\hat{E}^\varepsilon \hat{u}^\varepsilon \xrightarrow{L^2} \hat{E} \qquad \check{E}^\varepsilon \check{u}^\varepsilon \xrightarrow{L^2} \check{E}$$

$$\mathcal{J}^\varepsilon(\hat{u},\check{u}) = \varepsilon^{2s}\int_{\hat{\Omega}} \varepsilon^{t}\,\mathring{T}\cdot\hat{E}^\varepsilon\hat{u} + \varepsilon^{q+t} K\,\hat{H}^\varepsilon\hat{u}\;\mathring{T}\cdot\hat{H}^\varepsilon\hat{u} + \frac{1}{2}\frac{1}{\varepsilon^{2\eta}}\mathbb{L}\,\hat{E}^\varepsilon\hat{u}\cdot\hat{E}^\varepsilon\hat{u}\,dx$$

$$+\int_{\check{\Omega}}\frac{1}{2}\,\mathbb{C}\,\check{E}^\varepsilon\check{u}\cdot\check{E}^\varepsilon\check{u}\,dx$$

$$E_{\alpha\beta} = (E\hat{u})_{\alpha\beta} \qquad \check{P}^\varepsilon\check{u}^\varepsilon = (\check{u}_1^\varepsilon, \check{u}_2^\varepsilon, \varepsilon\,\check{u}_3^\varepsilon) \xrightarrow{H^1} \check{u}$$

$$\hat{u}^\varepsilon(\cdot,\cdot,0) = \check{u}^\varepsilon(\cdot,\cdot,0)$$

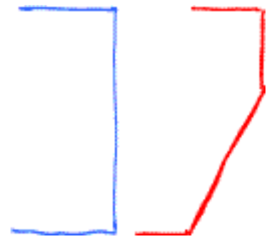
$\check{u}_\alpha(\cdot,\cdot,0)=0$	if $\eta>s$	$\check{u}_3(\cdot,\cdot,0)=0$	if $\eta>3s$
$\check{u}_\alpha(\cdot,\cdot,0)=\hat{u}_\alpha(\cdot,\cdot,0)$	if $\eta=s$	$\check{u}_3(\cdot,\cdot,0)=\hat{u}_3(\cdot,\cdot,0)$	if $\eta=3s$
$\hat{u}_\alpha(\cdot,\cdot,0)=0$	if $\eta<s$	$\hat{u}_3(\cdot,\cdot,0)=0$	if $\eta<3s$

if $\eta>3s$ $\quad \check{u}=0$ $\qquad \eta\le 3s$

if $\eta<s$ $\quad \hat{u}=0$ $\qquad \eta\ge s$

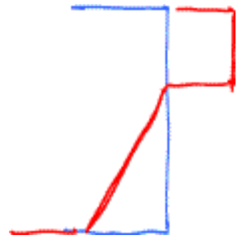
- $\eta = s$

$$\begin{cases} \hat{u}_\alpha = \xi_\alpha - \dfrac{\check{e}_u}{2} \xi_{3,\alpha} \\ \hat{u}_3 = 0 \end{cases} \qquad \begin{cases} \check{u}_\alpha = \xi_\alpha - \left(x_3 + \dfrac{\check{e}_u}{2}\right) \xi_{3,\alpha} \\ \check{u}_3 = \xi_3 \end{cases}$$



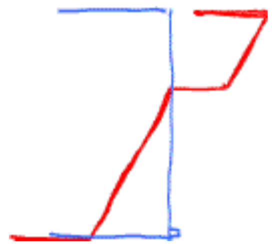
- $s < \eta < 3s$

$$\begin{cases} \hat{u}_\alpha = \xi_\alpha \\ \hat{u}_3 = 0 \end{cases} \qquad \begin{cases} \check{u}_\alpha = -x_3 \, \xi_{3,\alpha} \\ \check{u}_3 = \xi_3 \end{cases}$$



- $\eta = 3s$

$$\begin{cases} \hat{u}_\alpha = \xi_\alpha - \left(x_3 - \dfrac{\hat{e}_u}{2}\right) \xi_{3,\alpha} \\ \hat{u}_3 = \xi_3 \end{cases} \qquad \begin{cases} \check{u}_\alpha = -x_3 \, \xi_{3,\alpha} \\ \check{u}_3 = \xi_3 \end{cases}$$



either $\eta = s$ or $\eta = 3s$

"Pretraction Rigidity"

$$\frac{(\hat{H}^\varepsilon \hat{u}^\varepsilon)_{i\alpha}}{\varepsilon^{(\eta-s-q)/2}} \xrightarrow{L^2} \hat{H}_{i\alpha}$$

$$\frac{\hat{u}^\varepsilon_\alpha}{\varepsilon^{(\eta-s-q)/2}} \xrightarrow{L^2(0,1;H^1(\omega))} \hat{W}_\alpha \qquad \frac{\hat{u}^\varepsilon_3}{\varepsilon^{(\eta-s-q)/2}} \xrightarrow{H^1} \hat{W}_3$$

- $\eta = s \quad q \geq 2$
- $\eta = 3s \quad q \geq 2s+2$

$$\hat{H}_{\beta\alpha} = 0$$

$$\hat{H}_{3\alpha} = \begin{cases} \check{u}_{3,\alpha} & \text{if } q = 2 \\ 0 & \text{if } q > 2 \end{cases} \qquad \hat{H}_{3\alpha} = \begin{cases} \check{u}_{3,\alpha} & \text{if } q = 2s+2 \\ 0 & \text{if } q > 2s+2 \end{cases}$$

Variational convergence (Γ-convergence)

$\mathcal{J}^\varepsilon$ Γ-converges to $\mathcal{J}$ if:

1) $\forall\, (\hat{u}, \check{u}) \in \mathcal{A}_{\lim}$, $\forall\, (\hat{u}^\varepsilon, \check{u}^\varepsilon) \in \mathcal{A}$ s.t.

$$\frac{\hat{P}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \xrightarrow{H^1} \hat{u} \qquad \check{P}^\varepsilon \check{u}^\varepsilon \xrightarrow{H^1} \check{u}$$

we have

$$\liminf_{\varepsilon \to 0} \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) \geq \mathcal{J}(\hat{u}, \check{u})$$

2) $\forall\, (\hat{u}, \check{u}) \in \mathcal{A}_{\lim}$, $\exists\, (\hat{u}^\varepsilon, \check{u}^\varepsilon) \in \mathcal{A}$ s.t.

$$\frac{\hat{P}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \xrightarrow{H^1} \hat{u} \qquad \check{P}^\varepsilon \check{u}^\varepsilon \xrightarrow{H^1} \check{u}$$

and

$$\limsup_{\varepsilon \to 0} \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) \leq \mathcal{J}(\hat{u}, \check{u})$$

$$\eta = 3s$$

$$\mathcal{J}(\hat{u}, \check{u}) = \int_\omega \hat{h}\,\mathring{T}\cdot E\xi + \frac{1}{2}\,\hat{h}\,\mathbb{L}\,E\xi\cdot E\xi\,dx +$$

$$+ \int_\omega \tilde{\chi}_q\,\kappa\,\hat{h}\,\mathring{T}\,\overline{\nabla}\xi_3\cdot\overline{\nabla}\xi_3 + \left(\frac{\hat{h}^3}{12}\,\overline{\mathbb{L}} + \frac{\check{h}^3}{6}\,\overline{\mathbb{C}}\right)\overline{\nabla}^2\xi_3\cdot\overline{\nabla}^2\xi_3\,dx$$

$$\tilde{\chi}_q = \begin{cases} 1 & \text{if } q = 2s+2 \\ 0 & \text{if } q > 2s+2 \end{cases}$$

$$\xi_3 = 0$$

NO!

$$\eta = 8$$

$$\mathcal{J}(\hat{u}, \check{u}) = \int_{\omega} \hat{h}\,\mathring{T} \cdot \bar{E}\zeta + \frac{1}{2}\left(\hat{h}\,\bar{\mathbb{L}} + \check{h}\,\bar{\mathbb{C}}\right)\bar{E}\zeta \cdot \bar{E}\zeta \, dx \; +$$

$$+ \int_{\omega} - \frac{\hat{h}\check{h}}{2}\,\bar{\mathbb{L}}\,\bar{E}\zeta \cdot \bar{\nabla}^2 \zeta_3 \, dx \; +$$

$$+ \int_{\omega} - \frac{\hat{h}\check{h}}{2}\,\mathring{T} \cdot \bar{\nabla}^2 \zeta_3 + \chi_q\, \kappa\, h\,\mathring{T}\,\bar{\nabla}\zeta_3 \cdot \bar{\nabla}\zeta_3 + \left(\frac{\hat{h}\check{h}^2}{8}\,\bar{\mathbb{L}} + \frac{\check{h}^3}{12}\,\bar{\mathbb{C}}\right)\bar{\nabla}^2\zeta_3 \cdot \bar{\nabla}^2\zeta_3 \, dx$$

$$\chi_q = \begin{cases} 1 & \text{if } q = 2 \\ 0 & \text{if } q > 2 \end{cases}$$

Monoclinic materials $\begin{cases} \mathbb{L}_{\alpha\beta\gamma 3} = \mathbb{L}_{\alpha 333} = 0 \\ \mathbb{C}_{\alpha\beta\gamma 3} = \mathbb{C}_{\alpha 333} = 0 \end{cases}$

$\bar{A} \in \mathbb{R}^{2\times 2}_{sym}$

$$\bar{\mathbb{L}}\, \bar{A} \cdot \bar{A} = \min_{b} \mathbb{L} \left(\begin{array}{c:c} \bar{A} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hdashline b_1 \; b_2 & b_3 \end{array} \right) \cdot \left(\begin{array}{c:c} \bar{A} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hdashline b_1 \; b_2 & b_3 \end{array} \right)$$

$$\bar{\mathbb{C}}\, \bar{A} \cdot \bar{A} = \min_{b} \mathbb{C} \left(\begin{array}{c:c} \bar{A} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hdashline b_1 \; b_2 & b_3 \end{array} \right) \cdot \left(\begin{array}{c:c} \bar{A} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hdashline b_1 \; b_2 & b_3 \end{array} \right)$$

$\eta = 8$

$$\mathcal{J}(\hat{u},\check{u}) = \int_\omega \hat{h}\,\mathring{\overline{T}} \cdot \overline{E}\xi + \frac{1}{2}\left(\hat{h}\,\overline{\mathbb{L}} + \check{h}\,\overline{\mathbb{C}}\right)\overline{E}\xi \cdot \overline{E}\xi \, dx \, +$$

$$+ \int_\omega -\frac{\hat{h}\check{h}}{2}\,\overline{\mathbb{L}}\,\overline{E}\xi \cdot \overline{\nabla}^2 \xi_3 \, dx \, +$$

$$+ \int_\omega -\frac{\hat{h}\check{h}}{2}\,\mathring{\overline{T}} \cdot \overline{\nabla}^2 \xi_3 + \chi_q \, \kappa \, h\,\mathring{\overline{T}}\,\overline{\nabla}\xi_3 \cdot \overline{\nabla}\xi_3 + \left(\frac{\hat{h}\check{h}^2}{8}\,\overline{\mathbb{L}} + \frac{\check{h}^3}{12}\,\overline{\mathbb{C}}\right)\overline{\nabla}^2\xi_3 \cdot \overline{\nabla}^2 \xi_3 \, dx$$

$$\chi_q = \begin{cases} 1 & \text{if } q = 2 \\ 0 & \text{if } q > 2 \end{cases}$$

Monoclinic materials $\begin{cases} \mathbb{L}_{\alpha\beta\gamma 3} = \mathbb{L}_{\alpha 333} = 0 \\ \mathbb{C}_{\alpha\beta\gamma 3} = \mathbb{C}_{\alpha 333} = 0 \end{cases}$

Liminf inequality

$$\mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) = \int_{\hat{\Omega}} \mathring{T} \cdot \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} + k \frac{\hat{H}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{(\eta - s - q)/2}} \mathring{T} \cdot \frac{\hat{H}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{(\eta - s - q)/2}} + \frac{1}{2} \mathbb{L} \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \cdot \frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \, dx$$

$$+ \int_{\check{\Omega}} \frac{1}{2} \mathbb{C} \check{E}^\varepsilon \check{u}^\varepsilon \cdot \check{E}^\varepsilon \check{u}^\varepsilon \, dx$$

$$\frac{\hat{E}^\varepsilon \hat{u}^\varepsilon}{\varepsilon^{\eta - s}} \xrightarrow{L^2} \hat{E} \qquad \check{E}^\varepsilon \check{u}^\varepsilon \xrightarrow{L^2} \check{E} \qquad \frac{(\hat{H}^\varepsilon \hat{u}^\varepsilon)_{i\alpha}}{\varepsilon^{(\eta - s - q)/2}} \xrightarrow{L^2} \hat{H}_{i\alpha}$$

$$\liminf_{\varepsilon \to 0} \mathcal{J}^\varepsilon(\hat{u}^\varepsilon, \check{u}^\varepsilon) \geq \int_{\hat{\Omega}} \mathring{T} \cdot \hat{E} + k \hat{H} \mathring{T} \cdot \hat{H} + \frac{1}{2} \mathbb{L} \hat{E} \cdot \hat{E} \, dx + \int_{\check{\Omega}} \frac{1}{2} \mathbb{C} \check{E} \cdot \check{E} \, dx$$

• $\eta = s \quad q \geq 2$

$$\hat{E}_{\alpha\beta} = (E \hat{u})_{\alpha\beta} \qquad \check{E}_{\alpha\beta} = (E \check{u})_{\alpha\beta}$$

if $q > 2$

$$\liminf_{\varepsilon\to 0} \mathcal{J}^{\varepsilon}(\hat{u}^{\varepsilon}, \check{u}^{\varepsilon}) \geq \int_{\hat{\Omega}} \mathring{T}\cdot\hat{E} + k\,\hat{H}\,\mathring{T}\cdot\hat{H} + \tfrac{1}{2}\,\mathbb{L}\hat{E}\cdot\hat{E}\,dx + \int_{\check{\Omega}} \tfrac{1}{2}\,\mathbb{C}\check{E}\cdot\check{E}\,dx$$

$$\hat{E}_{\alpha\beta} = (E\hat{u})_{\alpha\beta} \qquad\qquad \check{E}_{\alpha\beta} = (E\check{u})_{\alpha\beta}$$

$$\overline{\mathbb{L}}\,\bar{A}\cdot\bar{A} = \min_{b}\ \mathbb{L}\begin{pmatrix} \bar{A} & \begin{array}{c} b_1 \\ b_2 \end{array} \\ \begin{array}{cc} b_1 & b_2 \end{array} & b_3 \end{pmatrix} \cdot \begin{pmatrix} \bar{A} & \begin{array}{c} b_1 \\ b_2 \end{array} \\ \begin{array}{cc} b_1 & b_2 \end{array} & b_3 \end{pmatrix}$$

$$\bar{A} \in \mathbb{R}^{2\times 2}_{sym}$$

$$\overline{\mathbb{C}}\,\bar{A}\cdot\bar{A} = \min_{b}\ \mathbb{C}\begin{pmatrix} \bar{A} & \begin{array}{c} b_1 \\ b_2 \end{array} \\ \begin{array}{cc} b_1 & b_2 \end{array} & b_3 \end{pmatrix} \cdot \begin{pmatrix} \bar{A} & \begin{array}{c} b_1 \\ b_2 \end{array} \\ \begin{array}{cc} b_1 & b_2 \end{array} & b_3 \end{pmatrix}$$

$$\liminf_{\varepsilon\to 0} \mathcal{J}^{\varepsilon}(\hat{u}^{\varepsilon}, \check{u}^{\varepsilon}) \geq \int_{\hat{\Omega}} \mathring{T}\cdot\hat{E} + k\,\hat{H}\,\mathring{T}\cdot\hat{H} + \tfrac{1}{2}\,\mathbb{L}\hat{E}\cdot\hat{E}\,dx + \int_{\check{\Omega}} \tfrac{1}{2}\,\mathbb{C}\check{E}\cdot\check{E}\,dx$$

$$\geq \int_{\hat{\Omega}} \mathring{T}\cdot\hat{E} + k\,\hat{H}\,\mathring{T}\cdot\hat{H} + \tfrac{1}{2}\,\overline{\mathbb{L}}\,\overline{\hat{E}}\cdot\overline{\hat{E}}\,dx + \int_{\check{\Omega}} \tfrac{1}{2}\,\overline{\mathbb{C}}\,\overline{\check{E}}\cdot\overline{\check{E}}\,dx$$

Thank you