UNIVERSITÀ DI PISA

L'Approccio Frattale alla Meccanica non Locale

(The fractal approach to non-local mechanics)

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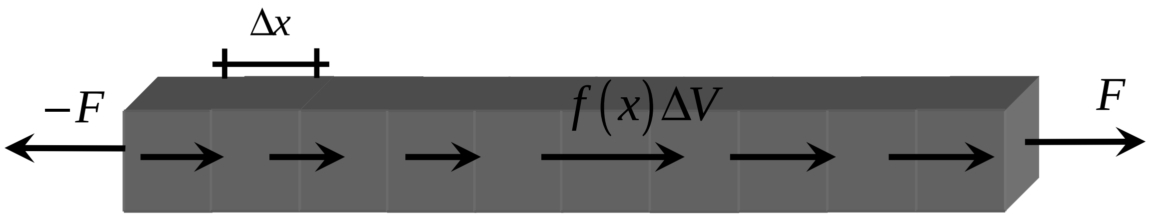
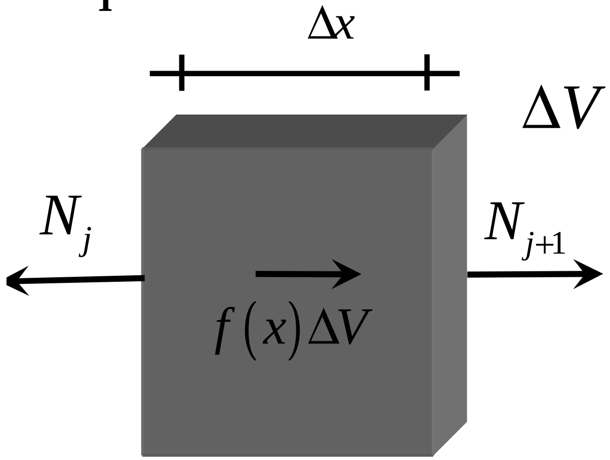
OUTLINES

- LATTICE THEORY
- GRADIENT AND STRONG NON-LOCAL THEORY
- PHYSICALLY-BASED APPROACH TO NON-LOCAL MECHANICS
- SELECTION OF THE DECAYING FUNCTION
- FRACTALS vs FRACTIONAL
- CONCLUSIONS

The Classical Continuum Mechanics (1D) Case



Equilibrium of solid element:



$$\Delta V = A\Delta x$$

$$N_{j+1} - N_j = f(x)\Delta V$$

$$\frac{\Delta N_j}{A} = -f(x)\Delta x \quad \xrightarrow{\Delta x \to 0} \quad \frac{d\sigma(x)}{dx} = -f(x)$$

Constitutive Equation (LOCAL)

$$\sigma = E\varepsilon = E\frac{du}{dx}$$

Governing equation of the 1D solid

$$\Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{f(x)}{E}$$

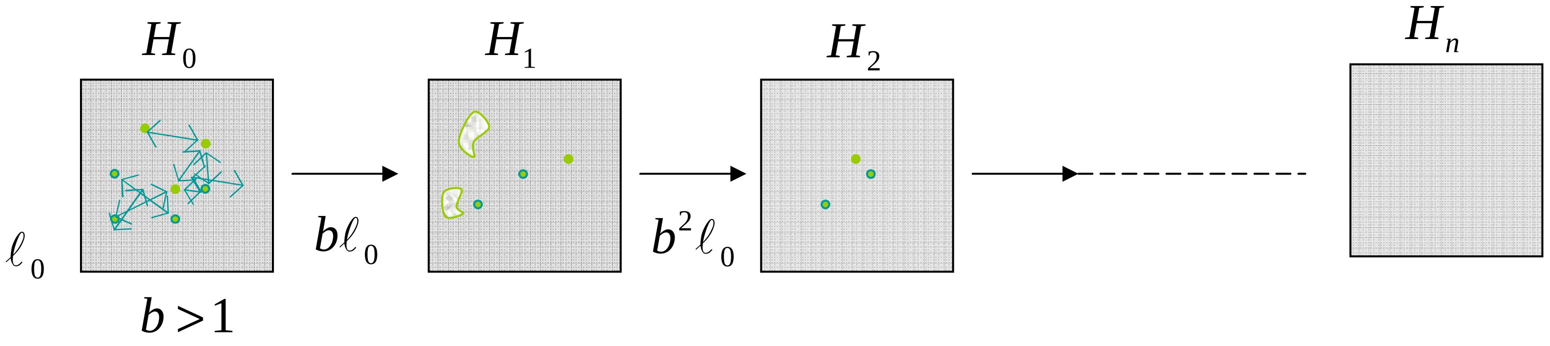
Boundary conditions:

$$EA\varepsilon_0 = -F_0 \quad ; \quad u(0) = u_0$$

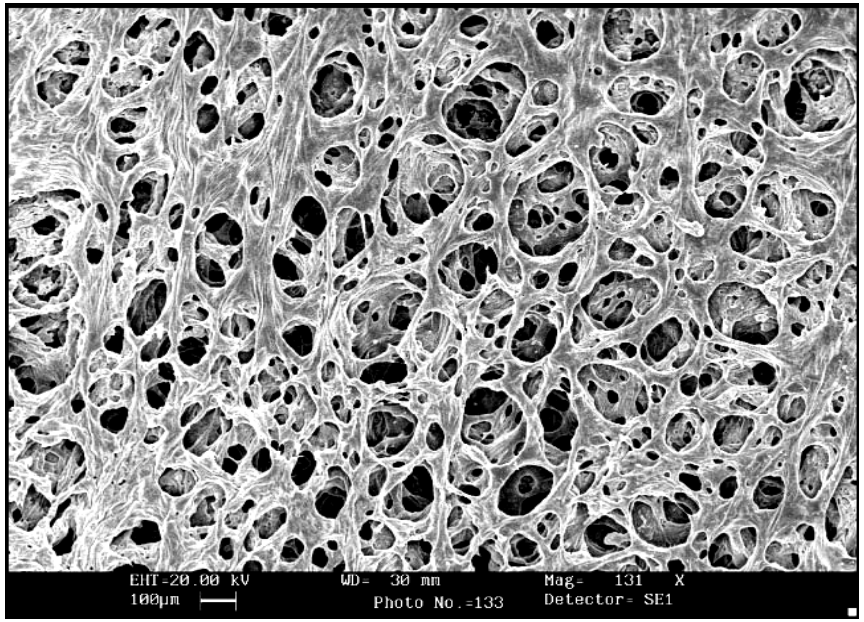
$$EA\varepsilon_L = F_L \quad ; \quad u(L) = u_L$$

The presence of microstructure in real-materials

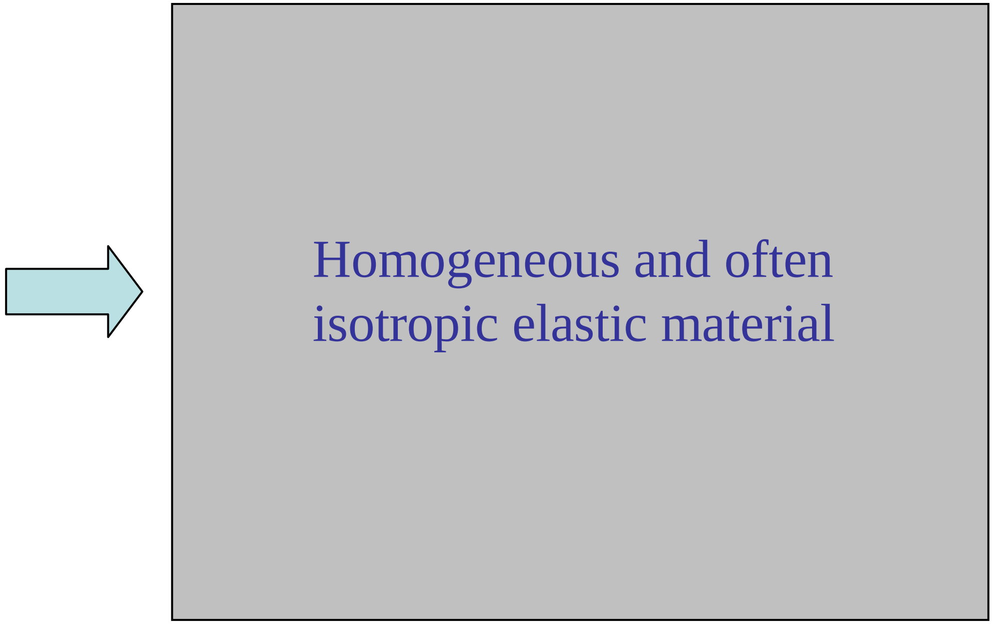
RENORMALIZATION (WILSON 1972)



Continuum Mechanics Approach



H_0

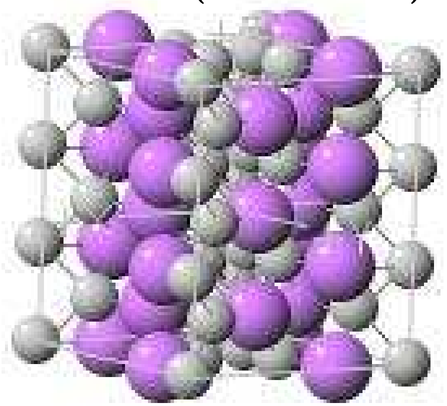


H_n

The Molecular Dynamics Approach

NANOSCALE

$O(1-10\ \ nm)$



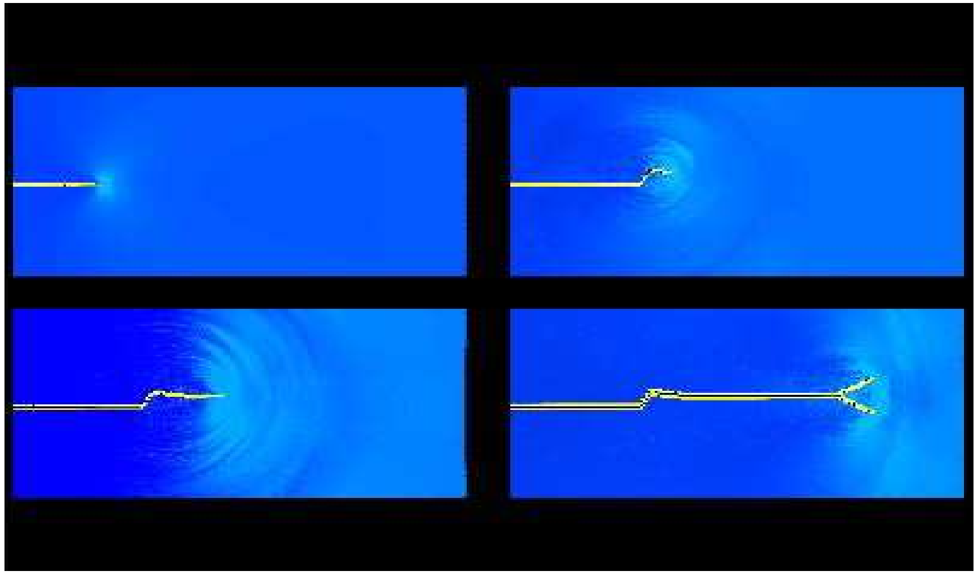
Each particle has three degree of freedom and its motion is ruled by the Newtons' law: $\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{F}(t)$

$\mathbf{u}(t)\in\Re^{3n}$ **Displacement vector**

$\mathbf{M}\in\Re^{3n\times 3n}$ **Mass matrix**

$\mathbf{K}\in\Re^{3n\times 3n}$ **Elastic bonds matrix**

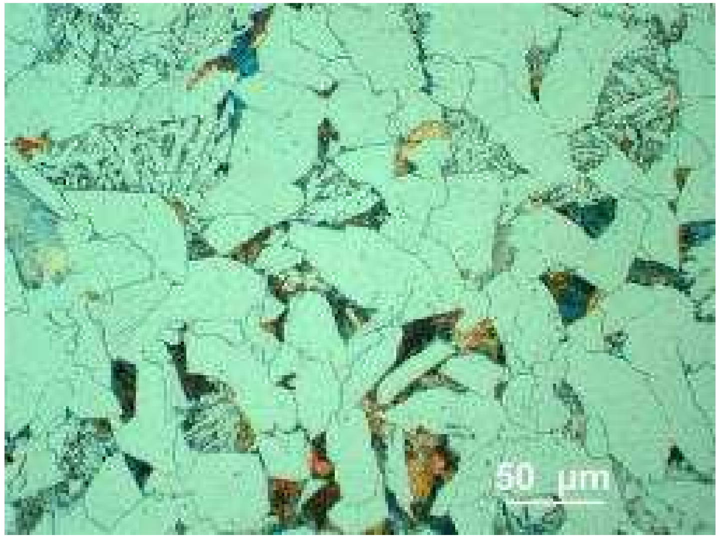
MESOSCALE $O(0.001-1\ \ \mu m)$

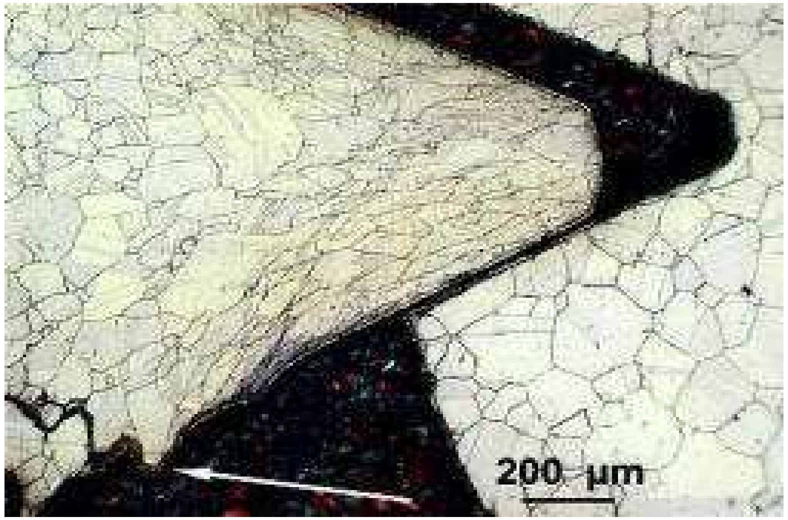


TERANUMBERS $O(10^{12})$ **DEGREE OF FREEDOM INVOLVED !!!**

The need for an enriched continuum mechanics

•In the late fifties and mid-sixties the basis of a generalized continuum mechanics had been proposed considering the inner microstructure

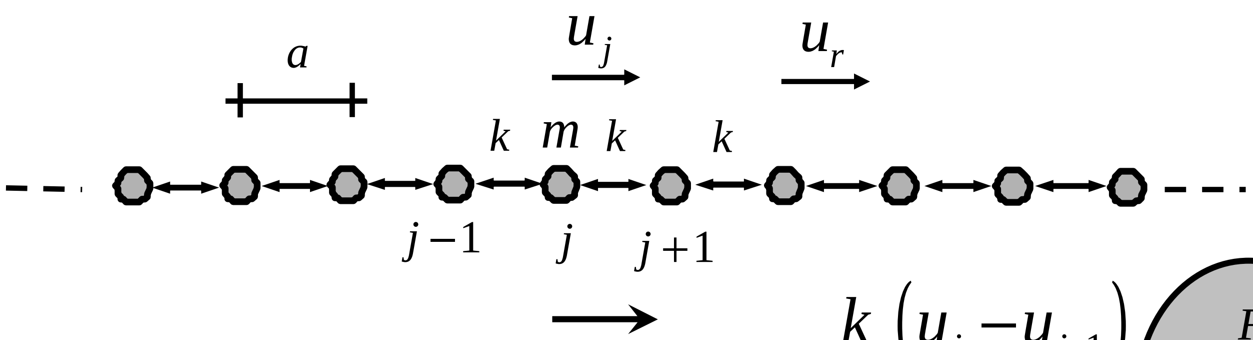


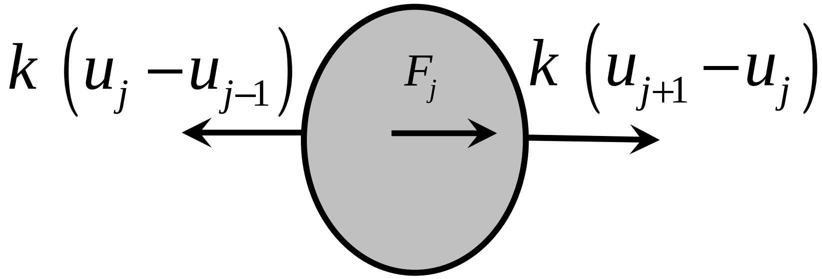


The theory of micromorphic continuum

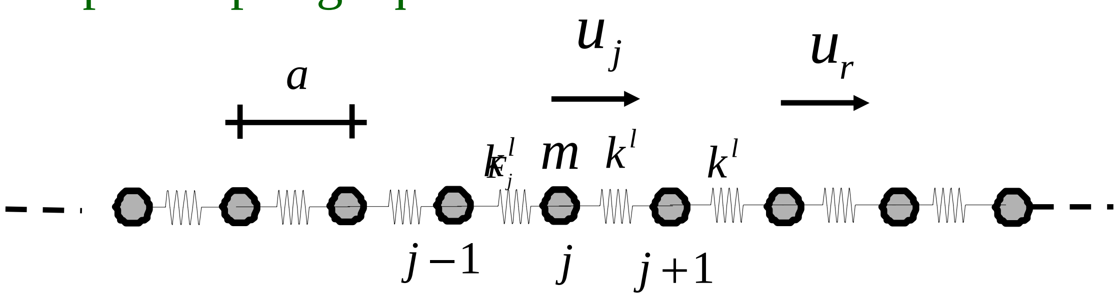
The Lattice Model of Materials (NN)

- Material properties often described at molecular level (Born-Von Karman):





- A point-spring equivalent model:



$$k^l\left(u_{j+1} - 2u_j + u_{j-1}\right) = -F_j$$

m - Mass of Lattice Atoms

k - Lattice Elastic Constant

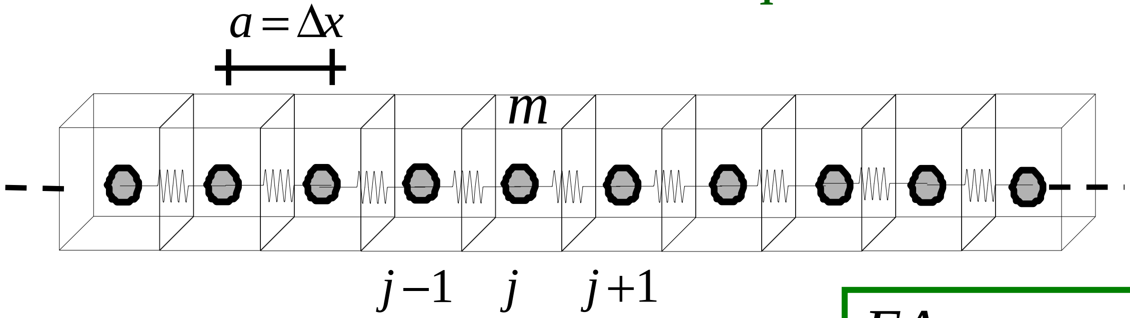
a - Lattice Distance

F_j = External load

$$k^l = k = EA/a$$

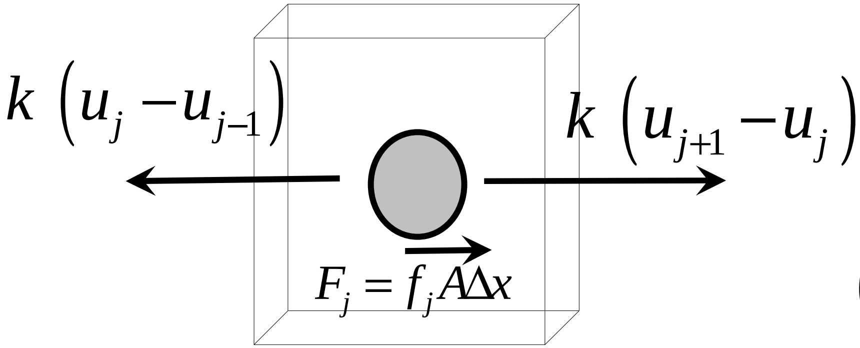
The Continuum Equivalence of the Lattice Models

- Derivation of the Waves Equation for an 1-D model



$f(x) =$ body force field

$$k = \frac{EA}{\Delta x} \; ; \; m = \rho A \Delta x$$



$$\frac{EA}{\Delta x}\left(u_{j+1} - 2u_j + u_{j-1}\right) = -f_j A \Delta x$$

$$\Delta x \rightarrow 0$$
$$\left(n \rightarrow \infty\right)$$

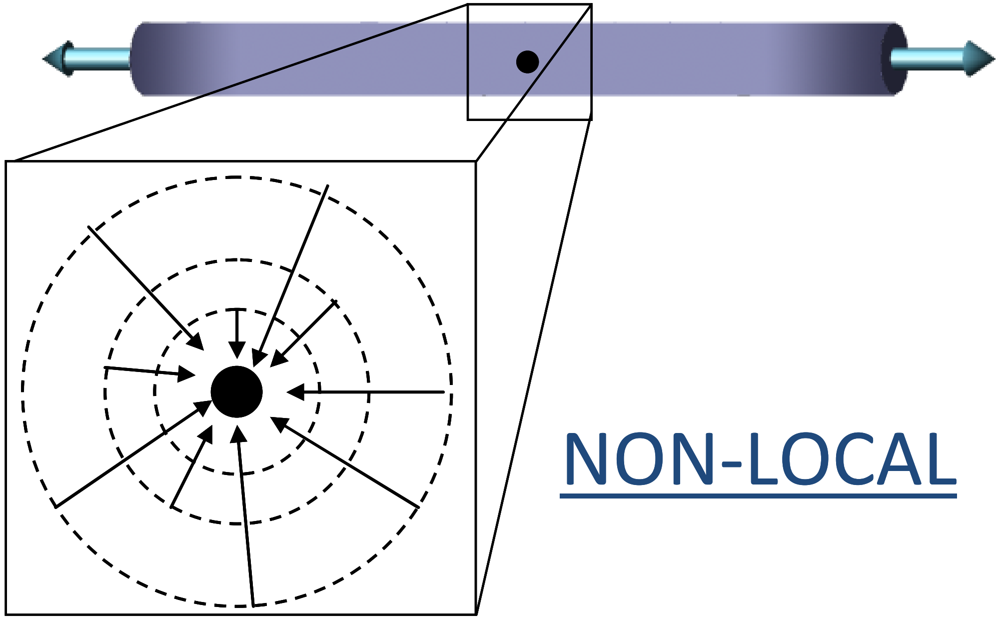
$$\frac{d^2u}{dx^2} = -\frac{f(x)}{E}$$

- As $\Delta x \rightarrow 0$ it is implicitely assumed that the same kind of interaction exists at each smaller scale (EUCLIDEAN OBJECT)

.

- Lattice elements may exchange interactions still at distance a (NN)

The Non-Local Elasticity Theories



Non-local model

GRADIENT (weak non-locality)

$$\sigma(x) = E_l\,\varepsilon(x) + E_1 \frac{d}{dx}\varepsilon(x) + E_2 \frac{d^2}{dx^2}\varepsilon(x) + ..$$

$\sigma(x)$	Axial stress
E_l, E_1, E_2	Elastic moduli
ε	Axial strain

INTEGRAL (strong non-locality)

$$\sigma(x) = E\varepsilon(x) + \int g(x,\xi)\varepsilon(\xi)\,d\xi$$

$g(x,\xi)$ Attenuation function

Essential references

Gradient non-local models

- Aifantis E.C., 1984, **On the microstructural origin of certain inelastic models**, *Trans. ASME, J. Eng. Mat. Trchn.*, Vol. 106, 326-330.
- Polizzotto C., Borino G., 1998, **A thermodynamics-based formulation of gradient-dependent plasticity**, *European Journal of Mechanics A/Solids*, Vol.17, 741-761.

Integral non-local models

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- Kröner E., 1967, **Elasticity theory of material with long range cohesive forces**, *International Journal of Solids and Structures*, Vol. 3, 731-742.
- Eringen A.C., Edelen D.G.B., 1972, **On nonlocal elasticity,** *International Journal Engineering Science*, Vol. 10, 233-248.

Fractal Mechanics

- Carpinteri A., 1994, **Fractal nature of material microstructure and size effects on apparent mechanical properties**, *Mechanics of Materials*, Vol.**18**, 89-101.
- Carpinteri A., Chiaia B., Cornetti P., 2001, **Static-Kinematic duality and the principle of virtual work in the mechanics of fractal media**, *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, 3-19.
- Carpinteri A., Cornetti P., 2002, **A fractional calculus approach to the description of stress and strain localization in fractal media**, *Chaos Solitons and Fractals*, Vol.13, 85-94.
- Epstein M., Sniatycki J., 2006, **Fractal Mechanics**, *Physica D*, Vol. 220, 54-68.

A Different approach

THE TEAM

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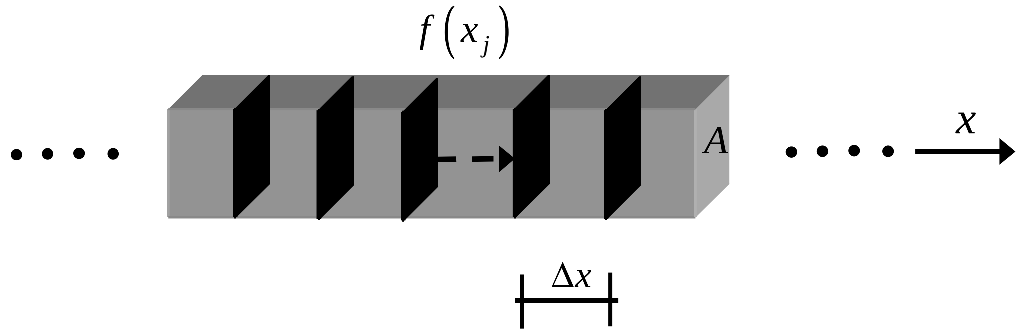
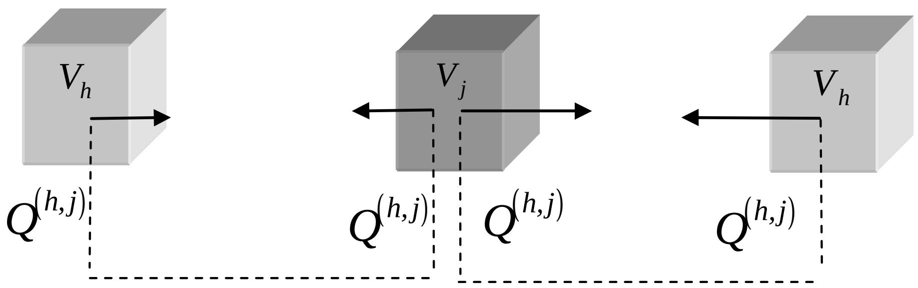
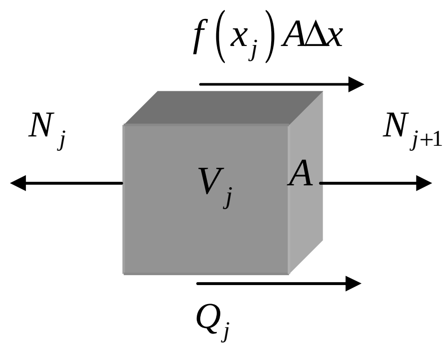
The proposed model (Bounded domain)







$$\left|Q^{(h,j)}\right| = \left|q^{(h,j)}\right| V_h V_j = \left|q^{(j,h)}\right| V_h V_j$$

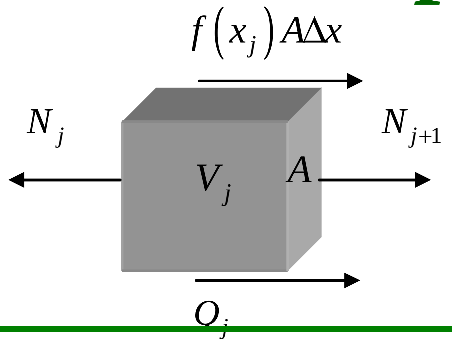
$$Q_j = \sum_{h=j+1}^{\infty} Q^{(h,j)} - \sum_{h=-\infty}^{j-1} Q^{(h,j)}$$

$$q^{(h,j)} = sign\left(x_h - x_j\right)\left(u_h - u_j\right) g\left(x_h, x_j\right)$$

Di Paola M., Zingales M., 2008, **Long-Range Cohesive Interactions of Non-Local Continuum Faced by Fractional Calculus,** ***International Journal of Solids and Structures, Vol.45, 5642-5659.***

Di Paola M., Failla G., Zingales M., 2009, **Physically-Based Approach to the Mechanics of Strong Non-Local Linear Elasticity Theory,** ***Journal of Elasticity, Vol. 97*, 103-130.**

The proposed model (continue…)



$$\sigma_l(x) = N(x)/A$$

$$\Delta N_j + Q_j + f(x_j)A\Delta x = \Delta N_j + \sum_{h=j+1}^{m} q^{(h,j)}(A\Delta x)^2 - \sum_{h=1}^{j-1} q^{(h,j)}(A\Delta x)^2 + f(x_j)A\Delta x = 0$$

$$E\frac{d^2u(x)}{dx^2} - A\int_0^L \left[u(x)-u(\xi)\right] g\left(\left|x-\xi\right|\right) d\xi = -f(x)$$

BOUNDED DOMAIN

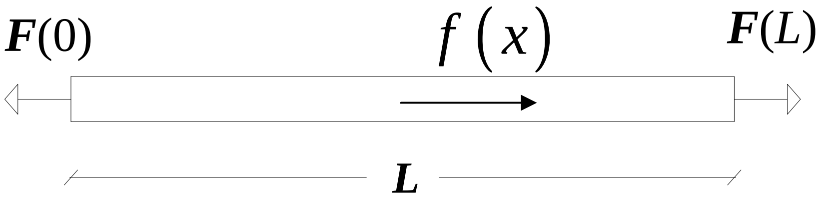
$$E\frac{d^2u(x)}{dx^2} - A\int_{-\infty}^{\infty} \left[u(x)-u(\xi)\right] g\left(\left|x-\xi\right|\right) d\xi = -f(x)$$

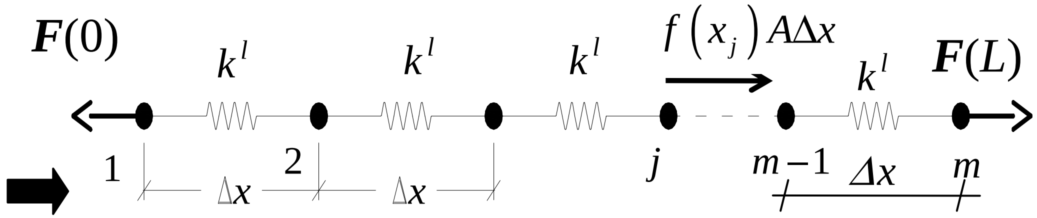
UNBOUNDED DOMAIN

Mechanical interpretation of non-locality

$$k^l = EA/\Delta x \qquad \Delta x = \frac{L}{m}$$

LOCAL MODEL





$$\sigma(x) = N(x)/A; \quad \varepsilon(x) = N(x)/EA$$

$$\boldsymbol{K}^l \boldsymbol{u} = \boldsymbol{f}$$

$$\mathbf{u}^T = [u_1 \quad u_2 \quad \dots \quad u_m]$$

$$\mathbf{f}^T = [f_1 \quad \dots \quad \dots \quad f_m]\Delta x$$

$$\mathbf{K}^l = \begin{bmatrix} K^l & -K^l & \dots & \dots & 0 \\ -K^l & 2K^l & -K^l & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -K^l & 2K^l & -K^l \\ 0 & \dots & \dots & -K^l & K^l \end{bmatrix}$$

Equilibrium of j[th] node

$$\lim_{\Delta x \to 0}\left[A\frac{\Delta^2 u(x_j)}{\Delta x} = -\frac{f_j A \Delta x}{E} \right] \Rightarrow \frac{d^2 u}{dx^2} = -\frac{f(x)}{E}$$

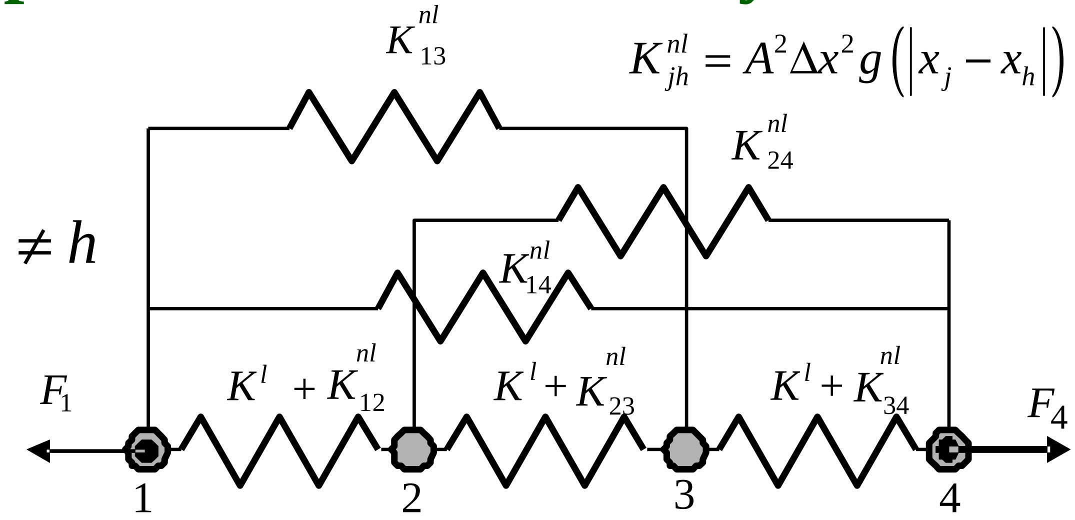
Mechanical interpretation of non-locality II

$$K_{jh}^{nl} = A^2 \Delta x^2 g\left(\left|x_j - x_h\right|\right)$$

NON-LOCAL MODEL

$$K_{jh}^{nl} = A^2 \Delta x^2 g\left(\left|x_j - x_h\right|\right)$$

$$j \neq h$$

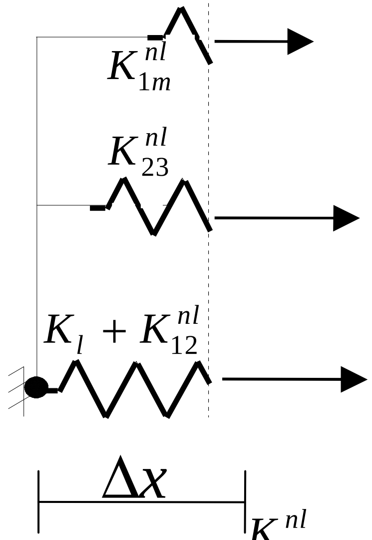


$$K_{jj}^{nl} = \sum_{\substack{h=1 \\ h \neq j}}^{m} K_{jh}^{nl}$$

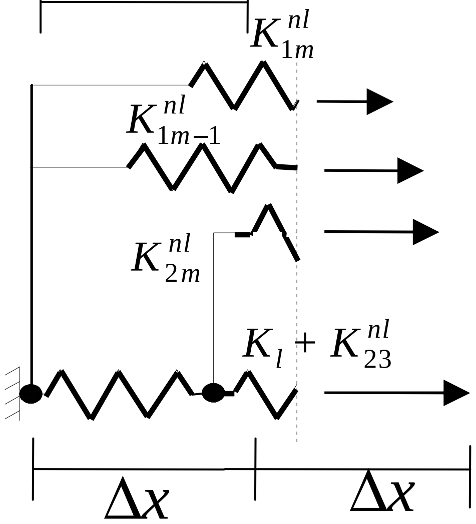
$$\boldsymbol{K} = \boldsymbol{K}^l + \boldsymbol{K}^{nl} \quad \Rightarrow \quad \boldsymbol{K}^{nl} = \begin{bmatrix} K_{11}^{nl} & -K_{12}^{nl} & -K_{13}^{nl} & \cdots & \cdots & -K_{1m}^{nl} \\ & K_{22}^{nl} & -K_{23}^{nl} & \cdots & \cdots & -K_{2m}^{nl} \\ & & \cdots & \cdots & \cdots & \cdots \\ & \text{SYM} & & & K_{m-1m-1}^{nl} & -K_{m-1\,m}^{nl} \\ & & & & & K_{mm}^{nl} \end{bmatrix}$$

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$$

The stress-strain Relations and the overall Cauchy stress



$$\sigma(x) = \frac{1}{A}\left(\sum_{j=2}^{m} K_{1j}^{nl}\left(u_j - u_1\right) + K_l\left(u_2 - u_1\right)\right) \qquad 0 < x < \Delta x$$



$$\sigma(x) = \frac{1}{A}\left(\sum_{j=3}^{m} K_{1j}^{nl}\left(u_j - u_1\right) + \sum_{j=2}^{m} K_{2j}^{nl}\left(u_j - u_2\right) + K_l\left(u_3 - u_2\right)\right)$$

$$\Delta x < x < 2\Delta x$$

GENERALIZING

$$r\Delta x < x < (r+1)\Delta x$$

$$\sigma(x) = \frac{1}{A}\left(\sum_{j=r+1}^{m} \sum_{h=1}^{r} K_{hj}^{nl}\left(u_j - u_h\right) + K_l\left(u_{r+1} - u_r\right)\right) \qquad \Delta x \to 0$$

$$= \left(E\frac{\left(u_r - u_{r-1}\right)}{\Delta x} - A\sum_{j=r+1}^{m} \sum_{h=1}^{r}\left(u_h - u_j\right) g\left(\left|x_j - x_h\right|\right)(\Delta x)^2\right)$$

The stress-strain relations and the overall Cauchy stress (II)

$$\sigma(x)=\frac{1}{A}\left(\sum_{j=r+1}^{m}\sum_{h=1}^{r}K_{hj}^{nl}\left(u_j-u_h\right)+K_l\left(u_{r+1}-u_r\right)\right) \qquad r\Delta x<x<(r+1)\Delta x$$

$$=\left(E\frac{\left(u_r-u_{r-1}\right)}{\Delta x}-A\sum_{j=r+1}^{m}\sum_{h=1}^{r}\left(u_h-u_j\right)g\left(\left|x_j-x_h\right|\right)(\Delta x)^2\right)$$

AT THE LIMIT

$$\Delta x\rightarrow 0$$

$$\sigma(x)=E\frac{du}{dx}-A\int_{\xi_2:x}^{L}\int_{\xi_1:0}^{x}\left(u(\xi_1)-u(\xi_2)\right)g\left(\left|\xi_1-\xi_2\right|\right)d\xi_1 d\xi_2$$

$$\sigma(x)=\sigma_l(x)+\sigma_{nl}(x)=\frac{1}{A}\left(N(x)+Q(x)\right)$$

$$\sigma_l(x)=\frac{N(x)}{A}=E\frac{du}{dx}$$

$$\sigma_{nl}(x)=\frac{Q(x)}{A}=-A\int_{\xi_2:x}^{L}\int_{\xi_1:0}^{x}\left(u(\xi_1)-u(\xi_2)\right)g\left(\left|\xi_1-\xi_2\right|\right)d\xi_1 d\xi_2$$

Comparisons between the Eringen model and the proposed model of long-range interactions

$$\sigma(x) = E\varepsilon(x) + C\int_a^b \varepsilon(\xi)\, g(|x-\xi|)\, d\xi$$

ERINGEN (1972)

$$\sigma(x) = E\varepsilon(x) - A\int_{\xi_1=x}^{b}\int_{\xi_2=a}^{x}\left[u(\xi_2)-u(\xi_1)\right] g(|\xi_1-\xi_2|)\, d\xi_1 d\xi_2$$

Mechanically-based (2008)

UNBOUNDED DOMAINS

(POWER-LAW, HELMOLTZ)

$$g_K(|x_j - x_m|) = C\exp(-|x_j - x_m|/\lambda)$$

$$\sigma(x) = E\varepsilon(x) + C\lambda^2\int_{-\infty}^{\infty} A\,\varepsilon(\xi)\exp\left[-|x-\xi|/\lambda\right] d\xi$$

(M. Di Paola, G. Failla, M. Zingales, 2009, Phisically-Based Approach to the Mechanics of Strong Non-Local Elasticity, *Journal of Elasticity*, *Vol. 97*,103-130.)

Comparisons between the Gradient and the proposed model of long-range interactions

$$E\frac{d^2u(x)}{dx^2}-A\int_{-\infty}^{\infty}\left[u(x)-u(\xi)\right]g\left(\left|x-\xi\right|\right)d\xi=-f(x)$$

Taylor series expansion of $u(x)$ <u>about location *x*</u>

$$E\frac{d^2u(x)}{dx^2}-\sum_{j=1}^{\infty}r_{2j}\frac{d^{2j}u(x)}{dx^{2j}}=-f(x) \qquad u(x)\in C_{\infty}$$

$$r_{2j}=\frac{A}{2j!}\int_{-\infty}^{\infty}(\xi-x)^{2j}g\left(\left|x-\xi\right|\right)d\xi$$

Di Paola M., Marino F., Zingales M., **2009,** **<u>A Generalized Model of Elastic Foundation based on Long-Range Interactions: Integral and Fractional Model</u>,** ***International Journal of Solids and Structures, 46, 3124-3137.***

The elastic problem of the 1D Continuum with long-range forces

$$
\begin{cases}
\boxed{\sigma(x) = E\dfrac{du}{dx} - A\displaystyle\int_{\xi_2:x}^{L}\int_{\xi_1:0}^{x}\left(u(\xi_1)-u(\xi_2)\right)g\left(\left|\xi_1-\xi_2\right|\right)d\xi_1 d\xi_2} \\
\dfrac{d\sigma(x)}{dx} = \dfrac{d}{dx}\left(\sigma_l(x)+\sigma_{nl}(x)\right) = -f(x) \\
\dfrac{du}{dx} = \varepsilon(x)
\end{cases}
$$

Constitutive

Equilibrium

Compatibility

BOUNDARY CONDITIONS

$$u(0) = u_0 \quad u(L) = u_L$$

Kinematic

$$
\boxed{
\begin{aligned}
A\sigma(x)\big|_L &= A\left(\sigma_l(x)\big|_L + \sigma_{nl}(x)\big|_L\right) = N(x)\big|_L = F \\
A\sigma(x)\big|_0 &= A\left(\sigma_l(x)\big|_0 + \sigma_{nl}(x)\big|_0\right) = N(x)\big|_0 = -F
\end{aligned}}
$$

Static

Di Paola M., Pirrotta A., Zingales M., 2010, **Mechanically-based approach to non-local elasticity: Variational principles ,** ***International Journal of Solids and Structures,*** *Vol.47*, ***539-548.***

The Distance-Decaying function

$$E\,\frac{d^2u}{dx^2} - A\int_0^L \left(u(x)-u(\xi)\right) g\left(|x-\xi|\right) d\xi = f(x)$$

$$EA(0)\varepsilon(0) = A\sigma_l(0) = -F_0 \quad ; \quad EA(L)\varepsilon(L) = A\sigma_l(L) = F_L$$

Local Cauchy stress equilibrates the external tractions

- The decaying function must be symmetric and must belong to the class of monotonically decreasing function of the arguments as from lattice theory.

$$g(x,\xi) = g(\xi,x) = g\left(|x-\xi|\right)$$

- One-Dimensional Geometry (Euclidean) → { Helmoltz, Bi-Helmoltz } → Kröner-Eringen model for unbounded solid

The decaying function: The Fractional Problem

- Fractional Power-Law: $$g\left(\left|x-\xi\right|\right)=\frac{c_{\alpha}E}{\Gamma(1-\alpha)}\frac{1}{\left|x-\xi\right|^{1+\alpha}} \qquad 0\le\alpha\le 1$$

- The Fractional Differential Problem:

$$\frac{d^2u(x)}{dx^2}-c_{\alpha}\left[\left(\hat{\mathbf{D}}_{0^+}^{\alpha}u\right)(x)+\left(\hat{\mathbf{D}}_{L^-}^{\alpha}u\right)(x)\right]=-\frac{f(x)}{E}$$

- Integral Parts of Marchaud Fractional Derivative

$$\left(\hat{\mathbf{D}}_{a^+}^{\alpha}f\right)(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_a^x\frac{f(x)-f(\xi)}{(x-\xi)^{(1+\alpha)}}d\xi$$

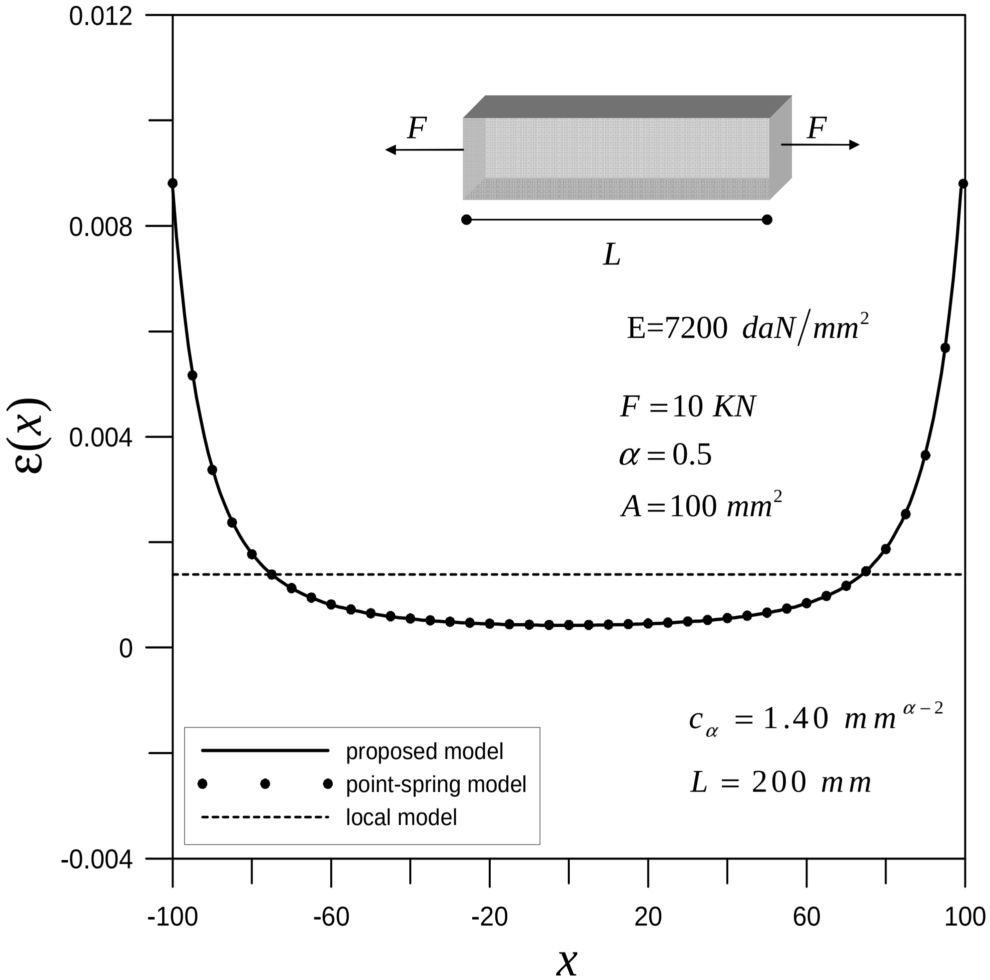
$$\left(\hat{\mathbf{D}}_{b^-}^{\alpha}f\right)(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_x^b\frac{f(x)-f(\xi)}{(\xi-x)^{(1+\alpha)}}d\xi$$

Unbounded Domains

$$\left(\mathbf{D}_{+}^{\alpha}f\right)(x)=\left(\hat{\mathbf{D}}_{+}^{\alpha}f\right)(x)\quad;\quad\left(\mathbf{D}_{-}^{\alpha}f\right)(x)=\left(\hat{\mathbf{D}}_{-}^{\alpha}f\right)(x) \qquad a\to-\infty, b\to\infty$$

Numerical application

Free-Free bar

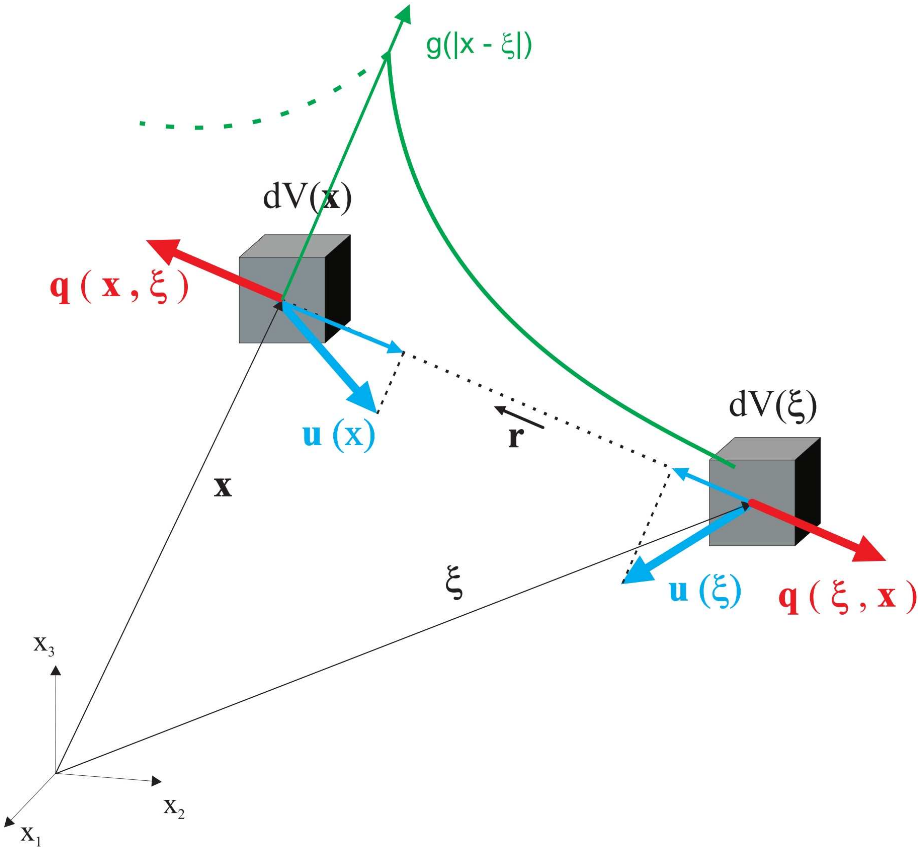


- In real materials the strains ARE NOT UNIFORM in tensile specimen under uniform stress

OK

$$\frac{d^2u(x)}{dx^2}-c_\alpha\left[\left(\hat{\mathbf{D}}^{\alpha}_{0^+}u\right)(x)+\left(\hat{\mathbf{D}}^{\alpha}_{L^-}u\right)(x)\right]=-\frac{f(x)}{E}$$

The 3D Non-Local Elasticity: The long-range forces



The relative displacement

$$\eta_k\left(\mathbf{x},\xi\right)=u_k\left(\mathbf{x}\right)-u_k\left(\xi\right)$$

The director vector

$$r_k\left(\mathbf{x},\xi\right)=\frac{x_k-\xi_k}{\sqrt{\left(x_k-\xi_k\right)\left(x_k-\xi_k\right)}}$$

The directional Jacobi tensor

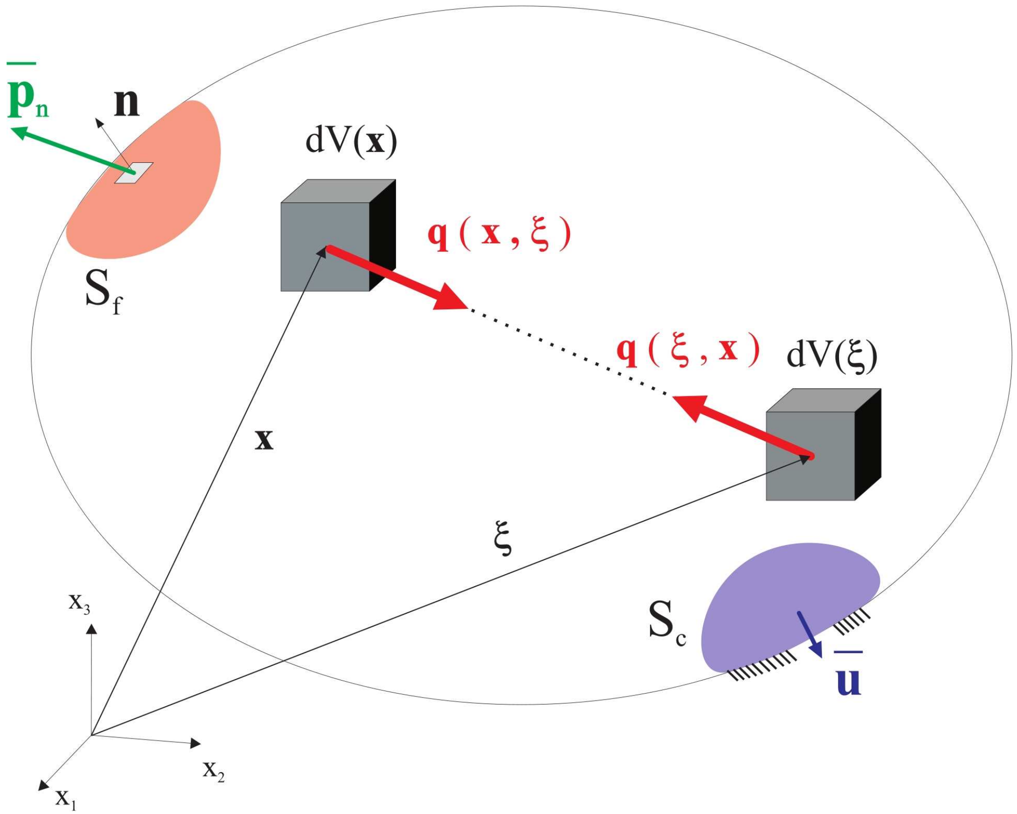
$$G_{jk}=r_k r_j g\left(\mathbf{x},\xi\right)$$

The specific long-range force applied in a point **x**

$$\mathbf{q}\left(\mathbf{x},\xi\right)=\mathbf{G}\left(\mathbf{x},\xi\right)\eta\left(\mathbf{x},\xi\right)$$

Di Paola M., Failla G., Zingales M., 2010, **The Mechanically-Based Approach to 3D Non-Local Elasticity Theory: Long-Range Central Interactions** , ***International Journal of Solids and Structures, doi: 10.1016/j.ijsolstr.2010.02.022.***

The 3D Non-Local Elastic Problem



Field Equations & boundary Conditions

$$\begin{cases}
\sigma^{(l)}_{kj,j}(\mathbf{x}) = -\bar{b}_k(\mathbf{x}) - f_k(\mathbf{x}) & \forall\ \mathbf{x} \in V \\
f_k(\mathbf{x}) = \int_V q_k(\mathbf{x},\xi)\, dV(\xi) = \int_V g_{kj}(\mathbf{x},\xi)\,\eta_j(\mathbf{x},\xi)\, dV(\xi) & \\
\varepsilon_{kj}(\mathbf{x}) = \frac{1}{2}\left(u_{k,j}(\mathbf{x}) + u_{j,k}(\mathbf{x})\right) & \forall\ \mathbf{x} \in V \\
\eta_k(\mathbf{x},\xi) = u_k(\xi) - u_k(\mathbf{x}) & \forall\ \mathbf{x},\xi \in V \\
\sigma^{(l)}_{kj}(\mathbf{x}) = 2\mu^* \varepsilon_{kj}(\mathbf{x}) + \delta_{kj}\lambda^* \varepsilon_{hh}(\mathbf{x}) & \forall\ \mathbf{x} \in V \\
q_k(\mathbf{x},\xi) = g_{kj}(\mathbf{x},\xi)\,\eta_j(\mathbf{x},\xi) & \forall\ \mathbf{x},\xi \in V
\end{cases}$$

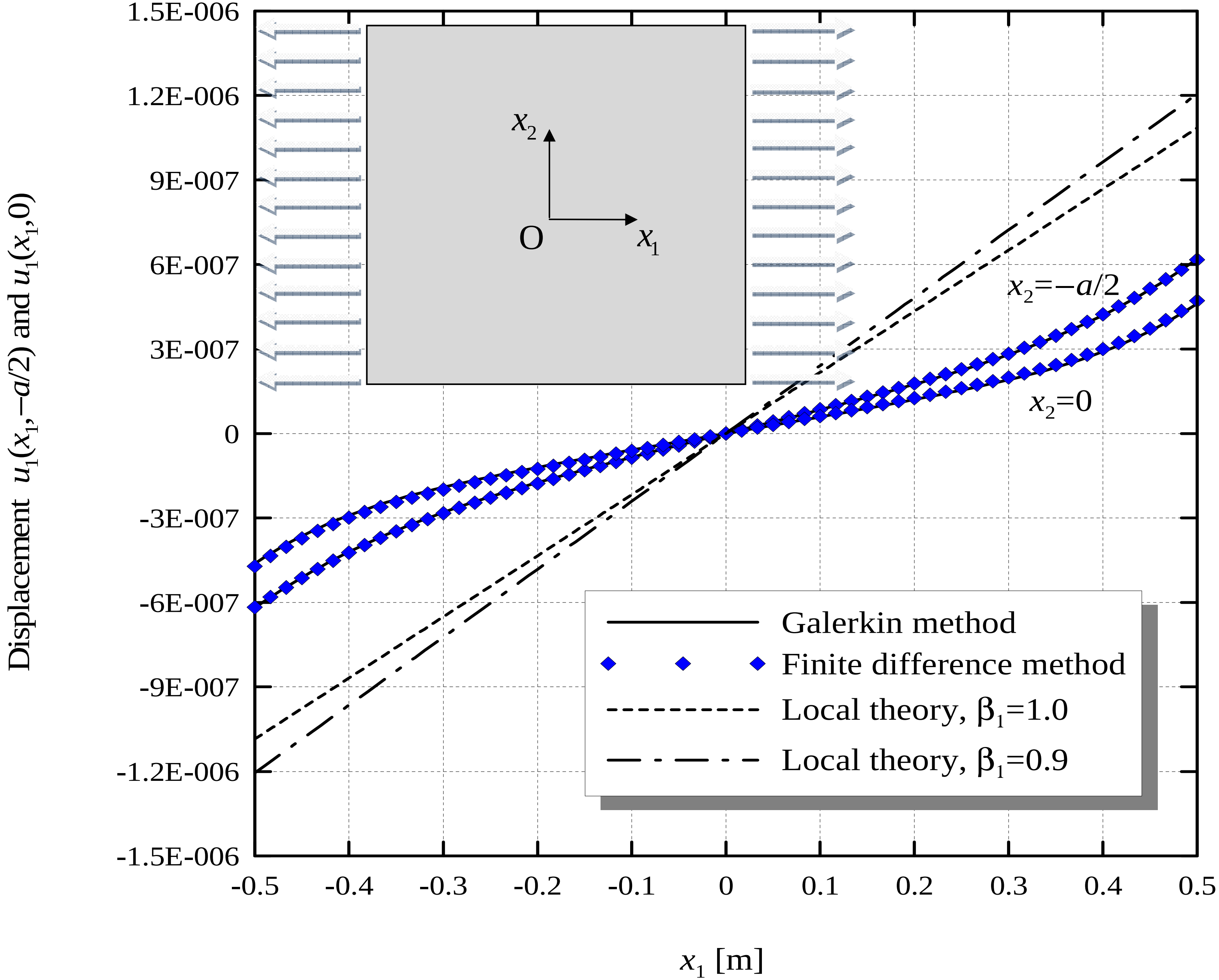
Euler-Lagrange Equations

$$\mu^* \nabla^2 u_k(\mathbf{x}) + \left(\lambda^* + \mu^*\right) u_{i,ik}(\mathbf{x})$$

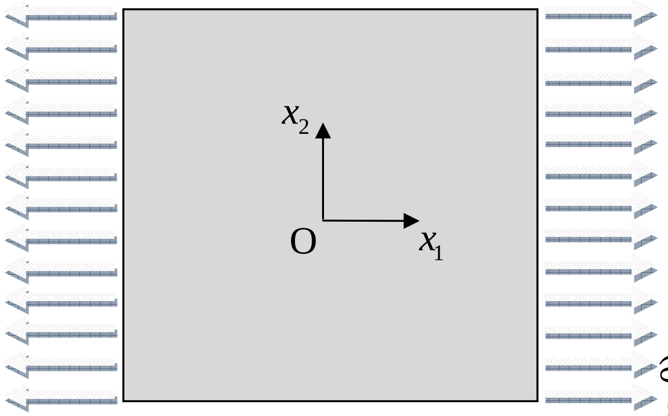
$$+ \int_V g_{kj}(\mathbf{x},\xi)\,\eta_j(\mathbf{x},\xi)\, dV(\xi) = -\bar{b}_k(\mathbf{x}) \quad \mathbf{x} \in V$$

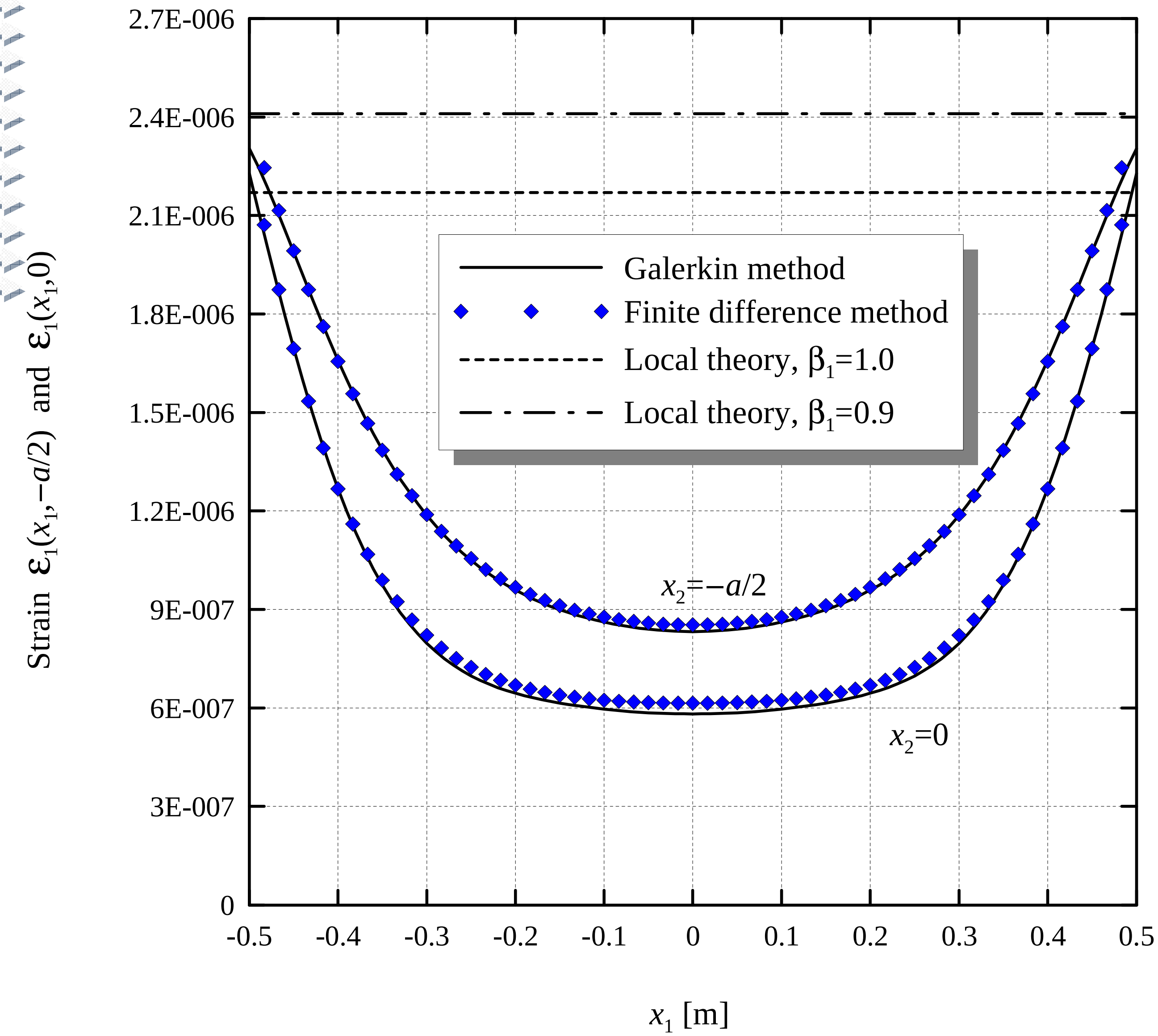
$$\begin{cases}
u_k(\mathbf{x}) = \bar{u}_k(\mathbf{x}) & \forall\ \mathbf{x} \in S_c \\
\sigma^{(l)}_{kj}(\mathbf{x})\, n_j = \bar{p}_{nk}(\mathbf{x}) & \forall\ \mathbf{x} \in S_f
\end{cases}$$

The 3D Non-Local Elastic Problem: Some Preliminar results



The 3D Non-Local Elastic Problem: Some Preliminar results





Research Developments

Di Paola M., Pirrotta A., Zingales M., **2010, Physically-Based approach to the mechanics of non-local continuum: Variational Principles**, ***International Journal of Solids and Structures*, 47, 539-548.** (Variational Formulation 1D)

Di Paola M., Failla G., Zingales M., **2009, Physically-Based approach to the mechanics of strong non-local linear elasticity**, ***Journal of Elasticity*, 97, *103-130*.** (Thermodynamic Consistency 1D)

Di Paola M., Marino F., Zingales M., **2009, A Generalized Model of Elastic Foundation based on Long-Range Interactions: Integral and Fractional Model**, ***International Journal of Solids and Structures*, 46, *3124-3137*.** (Non-Local elastic Foundations)

Failla G., Santini A., Zingales M., **2010**, **Solution Strategies for 1D Elastic Continuum with Long-Range Cohesive Interactions: Smooth and Fractional Decay,** ***Mechanics Research Communications*, 37, *13-21*.** (Approximate Solutions 1D)

Di Paola M., Failla G., Zingales M., **2010, The Mechanically-Based Approach to 3D Non-Local Elasticity Theory: Long-Range Central Interactions ,** ***International Journal of Solids and Structures*, doi: 10.1016/j.ijsolstr.2010.02.022.** (3D Linear Elasticity)

Zingales M., Di Paola M., Inzerillo G., **2009, The Finite Element Method for the Physically-Based Model of Non-Local Continuum**, ***International Journal for Numerical Methods in Engineering, (accepted)*** (FEM)

Zingales M., **2009, Waves Propagation in 1D Elastic Solids in presence of Long-Range Central Interactions**, ***Journal of Sound and Vibrations, (accepted).*** (1D propagation)

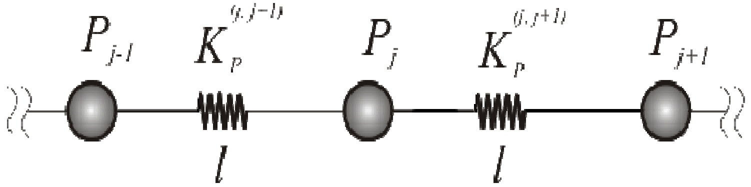
Cottone G., Di Paola M., Zingales M., **2009, Elastic Waves Propagation in 1D Continuum with Fractionally-decaying Long-Range Interactions**, ***Physica E, 42, 95-103.*** (1D propagation: Fractional calculus)

Carpinteri A., Cornetti P., Sapora A., Di Paola M. Zingales M., **2009, Fractional calculus in solid mechanics: local versus non-local approach** , ***Physica Scipta, T136, Article number* 014003** (1D Comparison)

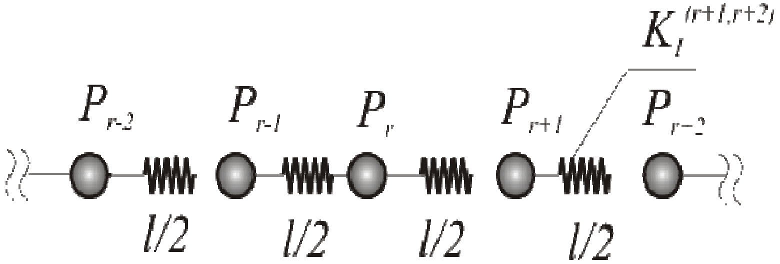
The Fractal Mechanical Model: The NN Lattice

- It corresponds to a mechanical, point-spring model as :

$$Q_P^{(j,j+1)} = K_P^{(j,j+1)}\left(u_{j+1}^{(P)} - u_j^{(P)}\right)$$

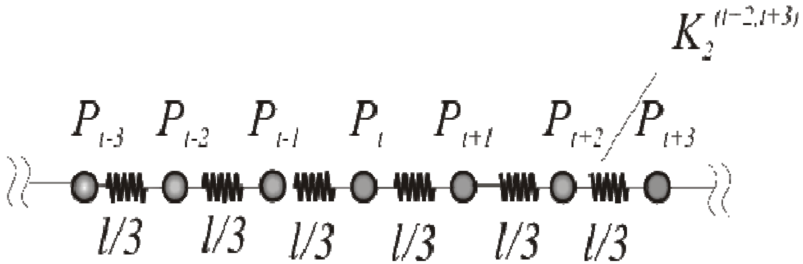


$$K_P^{(j,j+1)} = \frac{b_P A^2 l^2}{\Gamma(1-\beta)\left|x_{j+1} - x_j\right|^{\gamma}} = \frac{b_P \Delta V_P^2}{(l)^{\gamma}}$$



$$\Delta V_1 = Al / 2$$

$$K_1^{(j,j+1)} = \frac{2^{\gamma} b_P \Delta V_1^2}{(l)^{\gamma}} = 2^{\gamma-2} K_P^{(j,j+1)}$$



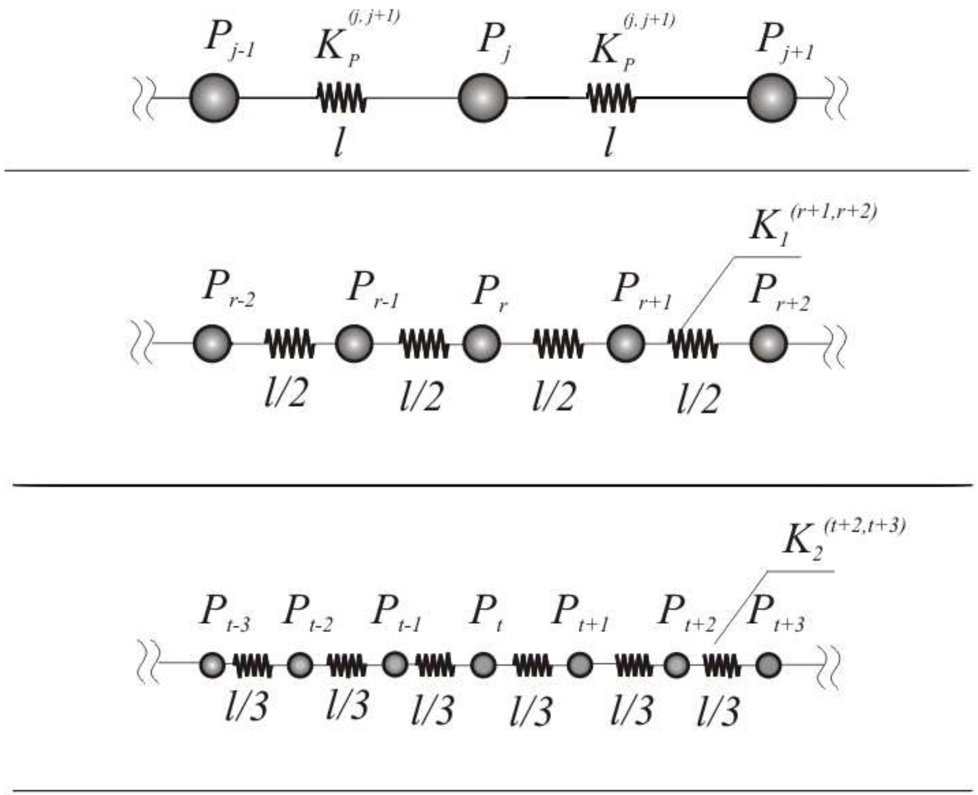
$$\Delta V_2 = Al/3$$

$$K_2^{(j,j+1)} = \frac{3^{\gamma} b_P \Delta V_2^2}{(l)^{\gamma}} = 3^{\gamma-2} K_P^{(j,j+1)}$$

.
.

$$K_h^{(j,j+1)} = h^{\gamma-2} K_P^{(j,j+1)}$$

The Fractal Mechanical Model: The HB Dimension



$$K_P^{(j,j+1)} = \frac{b_P \Delta V_1^2}{(l)^\gamma}$$

$$\Delta V_h = l / hA$$

$$K_h^{(j,j+1)} = \frac{b_P (\Delta V_h)^2}{(l/h)^\gamma} = h^{\gamma-2} K_P^{(j,j+1)}$$

$$\phi_h = \frac{1}{2} K_h^{(j,j+1)} \left(\frac{l}{h}\right)^2 = \frac{b_p A^2}{2} \left(\frac{l}{h}\right)^{4-\gamma}$$

- Elastic potential energy invariance at any observation scale

$$\Rightarrow \quad [\Phi_h]^s = h[\phi_h]^s = h\left(h^{\gamma-4}\right)^s [\Phi_0]^s = [\Phi_0]^s$$

$$d_H = \frac{1}{4-\gamma} \quad \Rightarrow \quad 0 < \gamma < 4$$

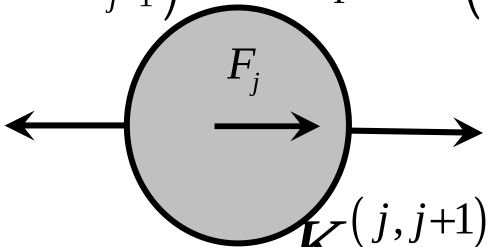
Di Paola M., Zingales M., 2009, **MultiScale Fractals and Fractional operators, A relevant example: Non-Local Elastic Continuum,** ***International Journal of Solids and Structures, (submitted).***

The Governing Operators

- Horizontal Equilibrium of the Generic Element of the NN lattice:

$$K_P^{(j,j-1)}\left(u_j^{(P)}-u_{j-1}^{(P)}\right) \qquad K_P^{(j,j+1)}\left(u_{j+1}^{(P)}-u_j^{(P)}\right)$$

$$\boxed{Q_P^{(j,j+1)}=K_P^{(j,j+1)}\left(u_{j+1}^{(P)}-u_j^{(P)}\right)}$$



$$K_h^{(j,j+1)}\left(u_{j+1}^{(h)}-2u_j^{(h)}+2u_{j-1}^{(h)}\right)=-f_jA\frac{l}{h}$$

Local Euclidean mechanics $\quad \gamma=3$

$$\boxed{b_pA\frac{\Delta^2u_j}{\Delta x^{\gamma-1}}=-f_j}$$

$\gamma\neq 3$??? $\quad$ Local Fractal mechanics

$$\boxed{b_pA\frac{d^2u}{dx^2}=-f(x)}$$

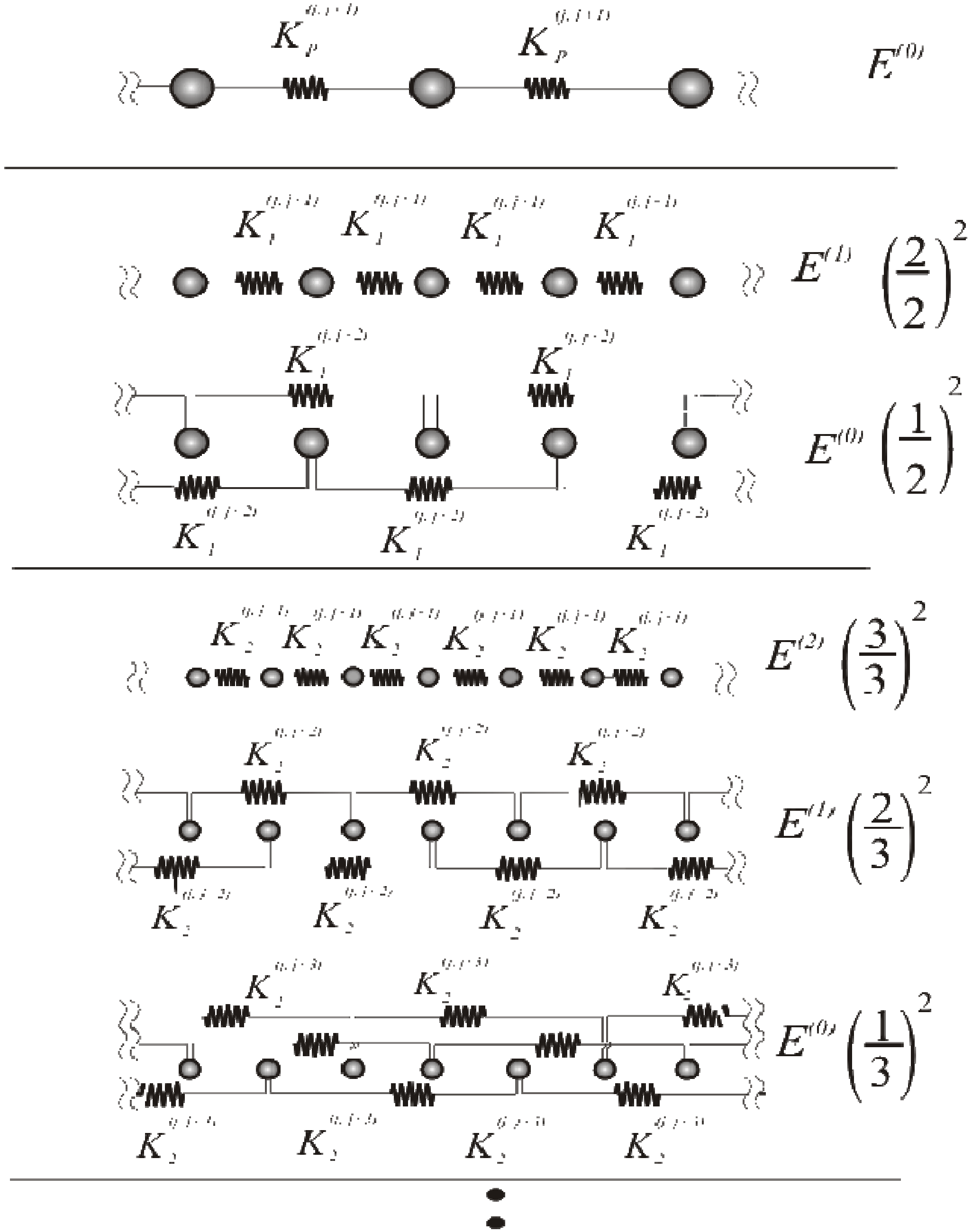
$$\Delta x=l/h$$

$$\boxed{b_pA\frac{\Delta^2u}{\Delta x^{\gamma-1}}=-f(x_j)}$$

- The classical govering equation of the continuum mechanics have been obtained without introducing contact, local, stress in the model. We argue that contact stress is obtained as the resultant of *short-range* forces between adjacent particles of solids.

The MultiScale Mechanical Fractal (MSF)

- The interaction distance do not change as we refine the observation scale.



$$K_P^{(j,j+1)} = \frac{b_P \Delta V_P^2}{(l)^{\gamma}}$$

$$K_1^{(j,j+1)} = 2^{\gamma-2} K_P^{(j,j+1)}$$

$$K_1^{(j,j+2)} = 2^{-2} K_P^{(j,j+1)}$$

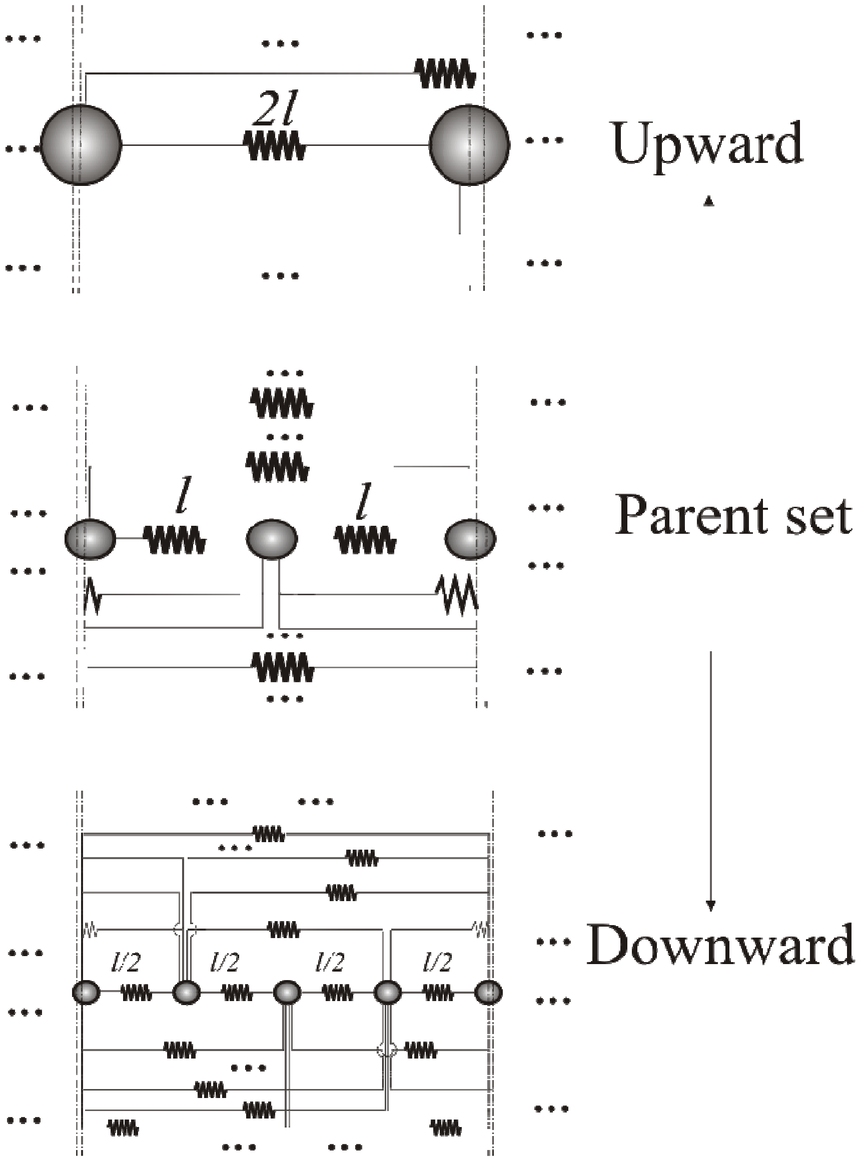
$$K_2^{(j,j+1)} = 3^{\gamma-2} K_P^{(j,j+1)}$$

$$K_2^{(j,j+2)} = \frac{3^{\gamma-2}}{2^{\gamma}} K_P^{(j,j+1)}$$

$$K_2^{(j,j+3)} = 3^{-2} K_P^{(j,j+1)}$$

The MultiScale Mechanical Fractal: The Scaling Law

- Physical interactions became negligible but cannot vanish beyond distance l so that they are <u>mathematically zero</u> only as $l \longrightarrow \infty$



The MSF is obtained as the union of self-similar elastic chains as $n \rightarrow \infty$

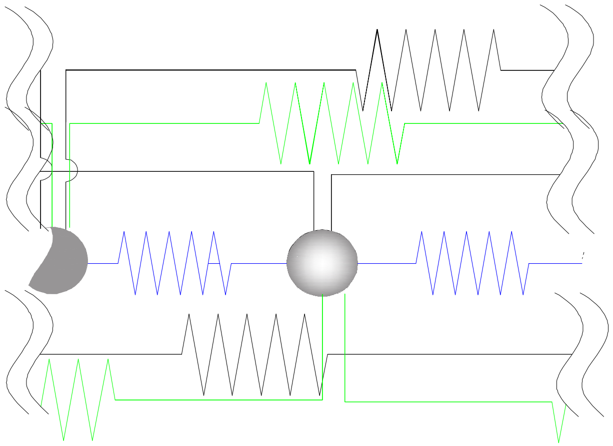
$$M_F = \lim_{n\rightarrow\infty} \bigcup_{j=1}^{n} p_j E^{(j)}$$

It maintains its self-similar nature at any resolution scale and the scaling law of the springs between different levels:

$$K_h^{(j,j+i)} = \frac{h^{\gamma-2}}{i^{\gamma}} \frac{b_p A^2 l^2}{l^{\gamma}} = \frac{h^{\gamma-2}}{i^{\gamma}} K_P^{(j,j+1)}$$

$$d_H = \frac{1}{4-\gamma} \Rightarrow 0 < \gamma < 4$$

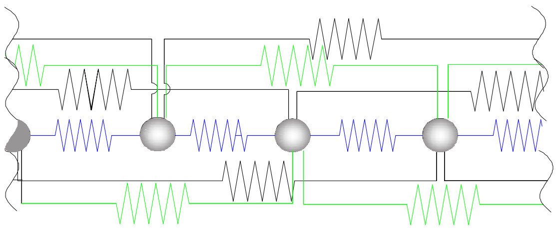
The MSF Model: ENERGY INVARIANCE



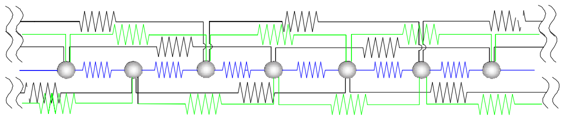
• The Elastic potential energy at the *h=1 scale*

$$\Phi_P = \frac{1}{2} K_P^{(j,j+1)} l^2 \qquad \Phi_1^s = \sum_{i=1}^{n} \left[\phi_0^{(j,j+i)} \right]^s = \sum_{i=1}^{n} \left[\frac{1}{i^{\gamma-2}} \Phi_P \right]^s$$

• The Elastic potential energy at the *r=1/n scale*



$$\Phi_h^s = \sum_{i=1}^{n} n \left[\phi_h^{(j,j+i)} \right] = \sum_{i=1}^{n} n \left[\frac{n^{\gamma-4}}{i^{\gamma-2}} \Phi_P \right]^s$$



THE INVARIANCE CONDITION

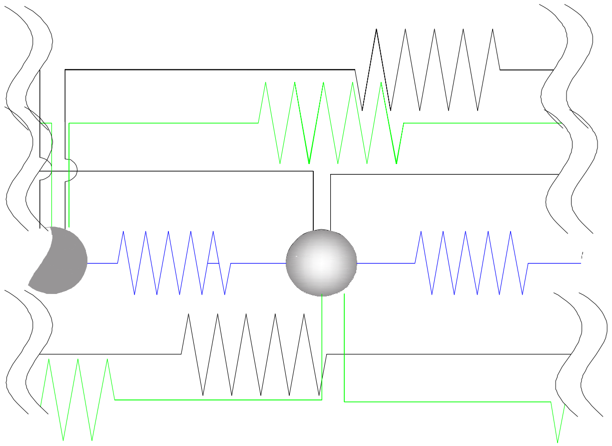
$$\Phi_1^s = \sum_{i=1}^{n} \left[\phi_0^{(j,j+i)} \right]^s = \sum_{i=1}^{n} \left[\frac{1}{i^{\gamma-2}} \Phi_P \right]^s = \sum_{i=1}^{n} n \left[\frac{n^{\gamma-4}}{i^{\gamma-2}} \Phi_P \right]^s = \sum_{i=1}^{n} n \left[\phi_h^{(j,j+i)} \right] = \Phi_h^s$$

The HB dimension of the mechanical MSF

$$d_H = \frac{1}{4-\gamma}$$

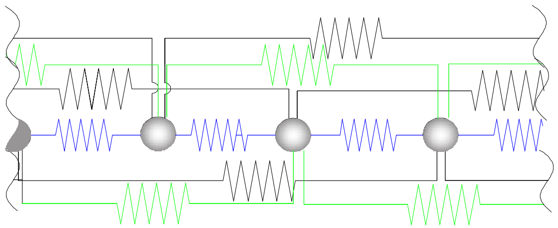
Q: What about operators ?

- Equilibrium equations at the n observation level: $F_j = f_j A \frac{l}{n}$



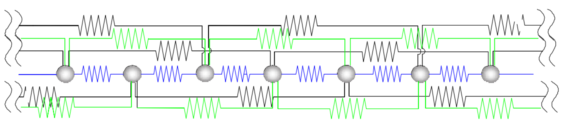
$$-\sum_{i=-\infty}^{j-1} K_n^{(j.j+i)}\left(u_j^{(n)} - u_{j+i}^{(n)}\right) + \sum_{p=r+1}^{\infty} K_n^{(j.j+i)}\left(u_{j+i}^{(n)} - u_j^{(n)}\right) = -F_j$$

$$K_n^{(j,j+i)} = \frac{b_p A^2}{(il/n)^{\gamma}}\left(\frac{l}{n}\right)^2$$



$$-\sum_{i=-\infty}^{j-1} \frac{b_P A^2\left(u_j^{(n)} - u_{j+i}^{(n)}\right)}{\left(x_j - x_{j+i}\right)^{\gamma}}\left(\frac{l}{n}\right)^2 + \sum_{i=j+1}^{\infty} \frac{b_P A^2\left(u_{j+i}^{(n)} - u_j^{(n)}\right)}{\left(x_{j+i} - x_j\right)^{\gamma}}\left(\frac{l}{n}\right)^2 = -f_j A\left(\frac{l}{n}\right)$$

$$\Delta x = l/n \rightarrow 0$$



$$b_P A\left[\int_{-\infty}^{x} \frac{u(x)-u(\xi)}{(x-\xi)^{\gamma}}d\xi + \int_{x}^{\infty} \frac{u(x)-u(\xi)}{(x-\xi)^{\gamma}}d\xi\right] = f(x)$$

A: Marchaud Derivatives !! $b_P A\left[\left(\mathbf{D}_+^{\bar{\alpha}}u\right)(x) + \left(\mathbf{D}_-^{\bar{\alpha}}u\right)(x)\right] = -f(x)$

Di Paola M., Zingales M., 2010, **The Multiscale Fractal Approach to Non-Local Elasticity: Direct Connection with Fractional Operators,** ***International Journal of Solids and Structures, (submitted).***

Q: What about operators ?
A: Marchaud fractional derivative

$$b_P A\left[\int_{-\infty}^{x}\frac{u(x)-u(\xi)}{(x-\xi)^{\gamma}}d\xi+\int_{x}^{\infty}\frac{u(x)-u(\xi)}{(x-\xi)^{\gamma}}d\xi\right]=f(x)$$

$$b_P=\frac{\alpha c_\alpha}{A\Gamma(1-\alpha)}$$

$$\gamma=1+\alpha$$

$$[c_\alpha]=FL^{\gamma-5}$$

$$\frac{\alpha c_\alpha}{\Gamma(1-\alpha)}\left[\int_{-\infty}^{x}\frac{u(x)-u(\xi)}{(x-\xi)^{1+\alpha}}d\xi+\int_{x}^{\infty}\frac{u(x)-u(\xi)}{(x-\xi)^{1+\alpha}}d\xi\right]=c_\alpha\left[\left(\mathbf{D}_{+}^{\alpha}u\right)(x)+\left(\mathbf{D}_{-}^{\alpha}u\right)(x)\right]=f(x)$$

$$d_H=\frac{1}{4-\gamma}=\frac{1}{3-\alpha}$$

$$0<\gamma<4 \Rightarrow \boxed{-1<\alpha<3}$$

Marchaud Fractional derivatives if: $0<\alpha\leq 2$

Fractional Riesz-Weyl potential if: $-1<\alpha\leq 0$

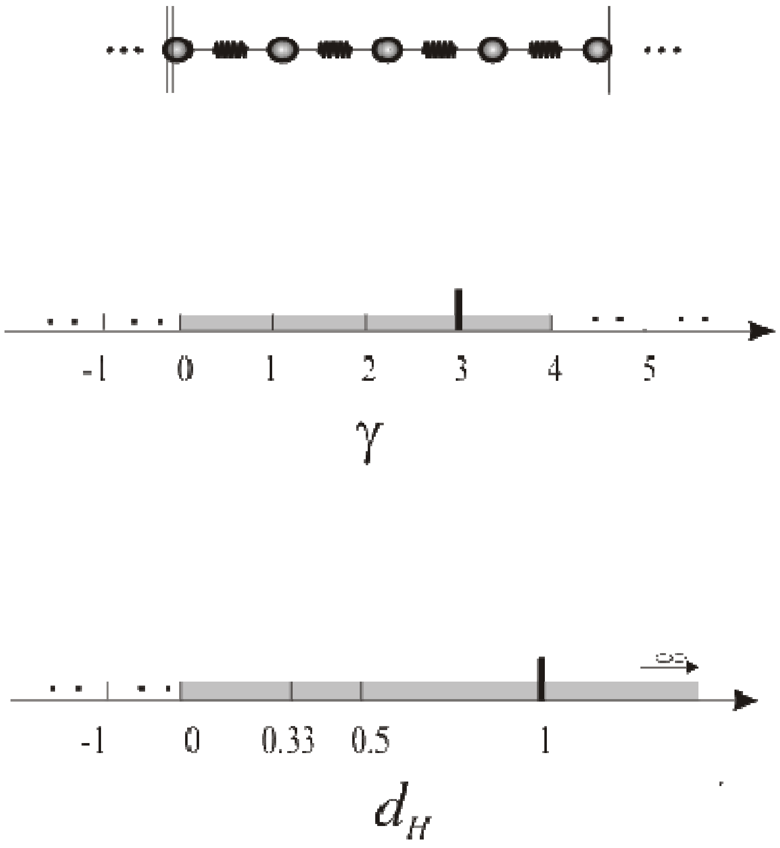
Non-admissible for internal stress scaling: $2<\alpha<3$

Di Paola M., Zingales M., 2009, **MultiScale Fractals and Fractional operators, A relevant example: Non-Local Elastic Continuum,** ***International Journal of Solids and Structures, (submitted).***

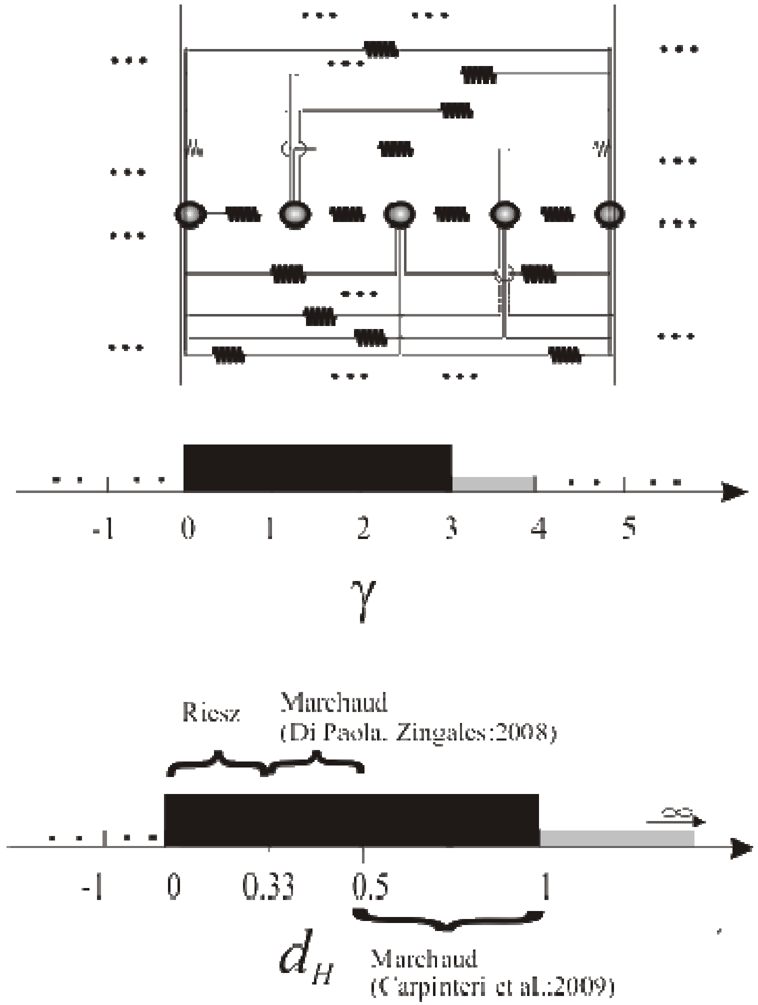
The role of the fractal dimension

$$d_H = \frac{1}{4-\gamma} \qquad 0 < \gamma < 4$$

Simple mechanical fractal: Euclidean solids only with classical differential operators



Multiscale mechanical fractals: Fractional-order operators.



The Euclidean case

$$d_H = \frac{1}{4-\gamma} = \frac{1}{3-\alpha}$$

$$\frac{\alpha c_\alpha}{\Gamma(1-\alpha)}\left[\int_{-\infty}^{x}\frac{u(x)-u(\xi)}{(x-\xi)^{1+\alpha}}d\xi+\int_{x}^{\infty}\frac{u(x)-u(\xi)}{(x-\xi)^{1+\alpha}}d\xi\right]=c_\alpha\left[\left(\mathbf{D}_+^\alpha u\right)(x)+\left(\mathbf{D}_-^\alpha u\right)(x)\right]=f(x)$$

$$\gamma = 3 \quad \Rightarrow \quad \begin{cases} d_H = 1 \\ \alpha = 2 \end{cases}$$

$$[c_\alpha] = FL^{\alpha-4}$$

$$c_2\left[\left(\mathbf{D}_+^2 u\right)(x)+\left(\mathbf{D}_-^2 u\right)(x)\right] = -2c_2\frac{d^2u}{dx^2} = f(x)$$

The Equilibrium Equation of Cauchy solid

$$E = 2c_2 \quad \Rightarrow \quad \boxed{\frac{d^2u}{dx^2} = -\frac{f(x)}{E}}$$

Conclusions

- Solid bodies with fractal mass distributions may be studied within the Mechanically-Based model of Long-Range Interactions and some important conclusions may be withdrawn.

1. The introduction of fractal distribution of the mass density in the solid leads to a fractal mechanical model represented by a point-spring model whose stiffness is power-law decreasing with the interdistance.

2. The assumption that only interactions with adjacent particle is included in the model leads toward a simple mechanical fractal that seems to be ruled by the local version of fractional operators. The order of the operators is connected with the fractal dimension of the mechanical fractal model (study in progress).

3. Assuming that long-range interactions are maintained at any resolution scale and that interactions extends to infinity a Multiscale Fractal mechanical model is obtained. The fractal dimension of the MSF coincides with that of the composing elastic chains.

4. The governing operators of the Multiscale Fractal mechanical model are Marchaud-type fractional differential equations. Therefore we conclude that such operators rules the physics of multiscale fractal sets.

THANK YOU FOR THE ATTENTION !