$\boxed{IS}$

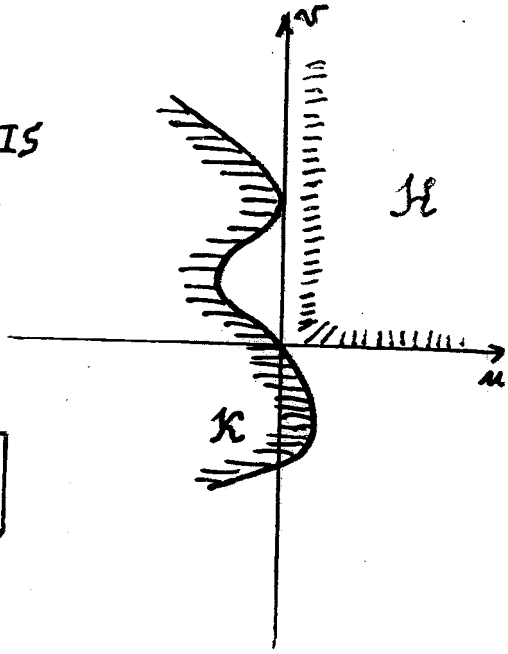
EXTREMUM PROBLEMS

$$\begin{cases} \min f(x) \\ g(x) \geq 0, \quad x \in X \end{cases}$$

$\mathcal{H} := \{(u,v) : u > 0, v \geq 0\} \subset IS$

$\mathcal{K} := \{(u,v) : u = f(\bar{x}) - f(x), \; v = g(x)\} \subseteq IS$

IMAGE SET



A FEASIBLE $\bar{x}$ IS GLOBAL MIN. POINT

$\Updownarrow$

$u := f(\bar{x}) - f(x) > 0$
$v := g(x) \geq 0, \quad x \in X$
IMPOSSIBLE

$\Updownarrow$

$\mathcal{H} \cap \mathcal{K}_{\bar{x}} = \emptyset$

$\Updownarrow$

$\mathcal{H} \cap \mathcal{E}_{\bar{x}} = \emptyset$

$\Uparrow$

$\exists \omega \in \Omega$ s.t.
$w(u,v;\omega) \leq 0$
$\forall (u,v) \in \mathcal{K}_{\bar{x}}$

$\Updownarrow$

$\exists \omega \in \Omega$ s.t.
$w(f(\bar{x}) - f(x), g(x); \omega) \leq 0, \; \forall x \in X$

$A_{\bar{x}}(x) = (f(\bar{x}) - f(x), g(x))$

$\mathcal{K}_{\bar{x}} = A_{\bar{x}}(X)$

$\mathcal{E}_{\bar{x}} := \mathcal{K} - \mathrm{cl}\,\mathcal{H}$

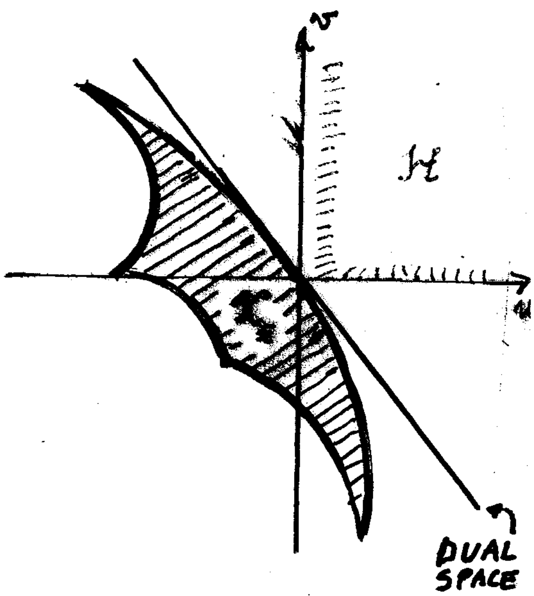
$w : IS \times \Omega \to \mathbb{R}$ s.t.

$\mathrm{lev}_{>0}\, w \supseteq \mathcal{H}$

WEAK SEPARATION FUNCTION

$w = \theta u + \langle \lambda, v \rangle$
LINEAR CASE

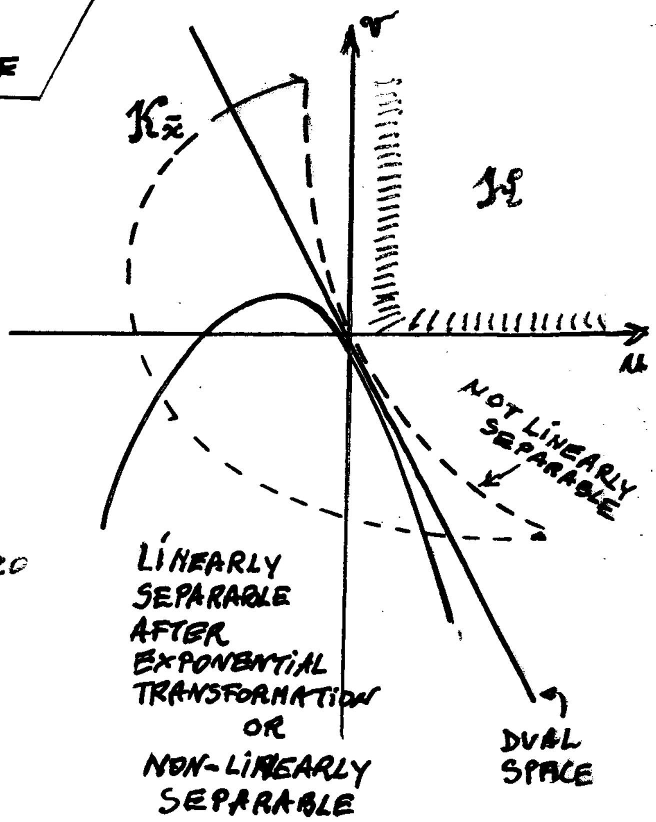
$w = \theta u + \sum_i \lambda_i v_i e^{-t_i v_i}$
EXPONENTIAL CASE



$$f(\bar{x}) \leq \overbrace{f(x) - \langle \lambda, g(x) \rangle}^{L(x;\lambda)}, \; \forall x \in X$$

$$L(\bar{x};\lambda) \leq L(\bar{x};\bar{\lambda}) \leq L(x;\bar{\lambda}), \; \forall x \in X, \forall \lambda \geq 0$$

SADDLE POINT CONDITION



$\boxed{\text{IS}}$

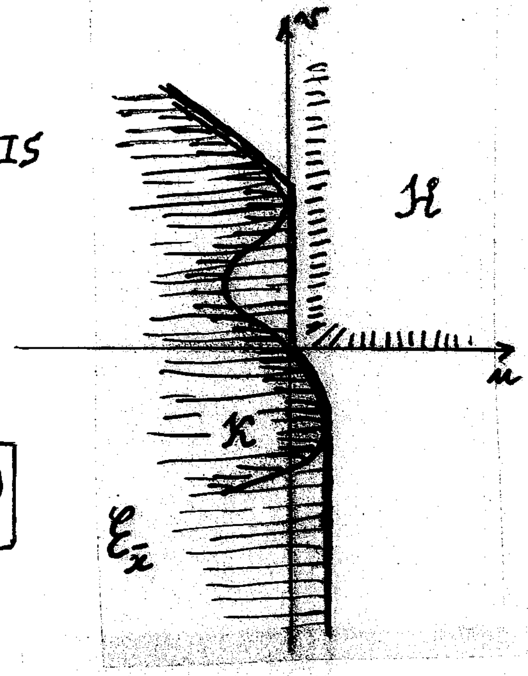
EXTREMUM PROBLEMS

$$\begin{cases} \min f(x) \\ g(x) \geq 0, \; x \in X \end{cases}$$

$\mathcal{H} := \{(u,v) : u > 0, v \geq 0\} \subset IS$

$\mathcal{K} := \{(u,v) : u = f(\bar{x}) - f(x), \; v = g(x)\} \subseteq IS$

IMAGE SET



$$\boxed{\begin{array}{l}\text{A FEASIBLE } \bar{x} \\ \text{IS GLOBAL MIN. POINT}\end{array}}$$

$$\Updownarrow$$

$$\begin{array}{r} u := \\ v := \\ \\ \end{array} \boxed{\begin{array}{l} f(\bar{x}) - f(x) > 0 \\ g(x) \geq 0, \; x \in X \\ \text{IMPOSSIBLE} \end{array}}$$

$$\Updownarrow$$

$$\boxed{\mathcal{H} \cap \mathcal{K}_{\bar{x}} = \emptyset}$$

$$\Updownarrow$$

$$\boxed{\mathcal{H} \cap \mathcal{E}_{\bar{x}} = \emptyset}$$

$$\Uparrow$$

$$\boxed{\begin{array}{l} \exists \omega \in \Omega \text{ s.t.} \\ w(u,v;\omega) \leq 0 \\ \forall (u,v) \in \mathcal{K}_{\bar{x}} \end{array}}$$

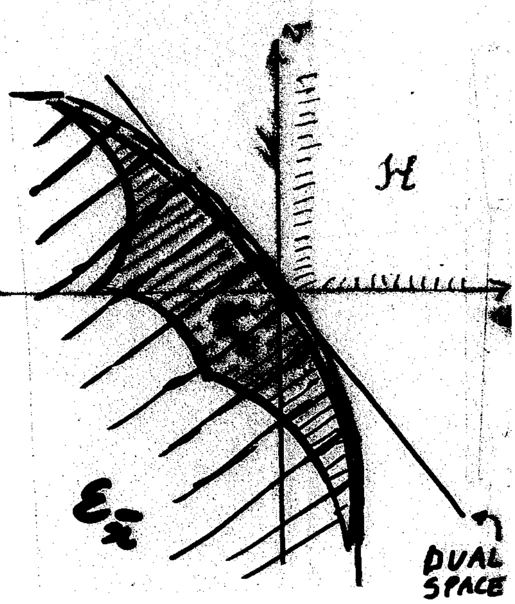
$$\Updownarrow$$

$$\boxed{\begin{array}{l} \exists \omega \in \Omega \text{ s.t.} \\ w(f(\bar{x}) - f(x), g(x); \omega) \leq 0, \; \forall x \in X \end{array}}$$

$$\boxed{\begin{array}{l} A_{\bar{x}}(x) = (f(\bar{x}) - f(x), g(x)) \\ \mathcal{K}_{\bar{x}} = A_{\bar{x}}(X) \\ \mathcal{E}_{\bar{x}} := \mathcal{K} - \operatorname{cl} \mathcal{H} \end{array}}$$

$$\boxed{\begin{array}{l} w : IS \times \Omega \to \mathbb{R} \\ \text{s.t.} \\ \operatorname{lev}_{>0} w \supseteq \mathcal{H} \\ \text{WEAK SEPARATION FUNCTION} \end{array}}$$

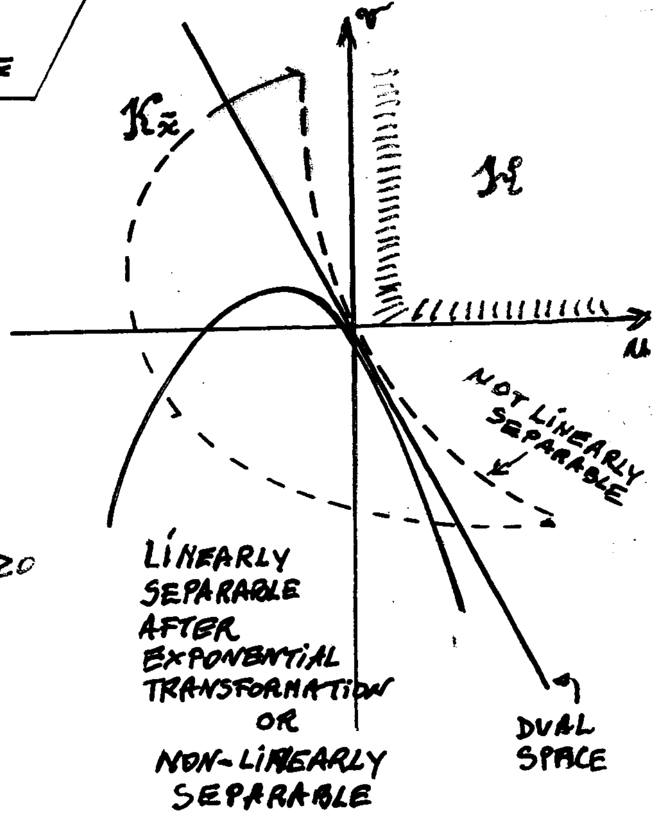
$$\boxed{\begin{array}{l} w = \theta u + \langle \lambda, v \rangle \\ \text{LINEAR CASE} \\ w = \theta u + \sum_i \lambda_i v_i e^{-t_i v_i} \\ \text{EXPONENTIAL CASE} \end{array}}$$



$$f(\bar{x}) \leq \overbrace{f(x) - \langle \lambda, g(x) \rangle}^{L(x;\lambda)}, \; \forall x \in X$$

$$L(\bar{x};\lambda) \leq L(\bar{x};\bar{\lambda}) \leq L(x;\bar{\lambda}), \; \forall x \in X, \; \forall \lambda \geq 0$$

SADDLE POINT CONDITION



$\boxed{IS}$

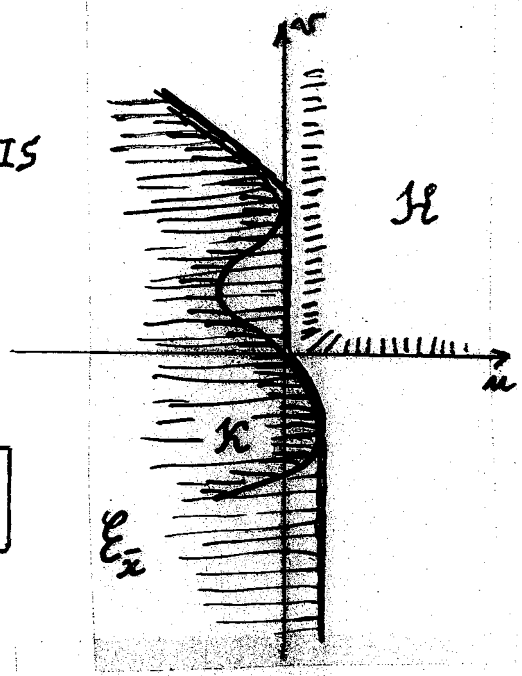
EXTREMUM PROBLEMS

$$\begin{cases}\min\ f(x)\\ g(x)\geq 0\,,\ x\in X\end{cases}$$

$\mathcal{H} := \{(u,v) : u>0, v\geq 0\}\subset IS$

$\mathcal{K} := \{(u,v) : u = f(\bar{x})-f(x),\ v = g(x)\}\subseteq IS$

IMAGE SET



A FEASIBLE $\bar{x}$ IS GLOBAL MIN. POINT

$\Updownarrow$

$u := f(\bar{x}) - f(x) > 0$

$v := g(x) \geq 0\,,\ x\in X$

IMPOSSIBLE

$\Updownarrow$

$\mathcal{H}\cap\mathcal{K}_{\bar{x}} = \emptyset$

$\Updownarrow$

$\mathcal{H}\cap\mathcal{E}_{\bar{x}} = \emptyset$

$\Uparrow$

$\exists\,\omega\in\Omega$ s.t. $w(u,v;\omega)\leq 0$ $\forall (u,v)\in\mathcal{K}_{\bar{x}}$

$\Updownarrow$

$\exists\,\omega\in\Omega$ s.t. $w(f(\bar{x})-f(x), g(x);\omega)\leq 0,\ \forall x\in X$

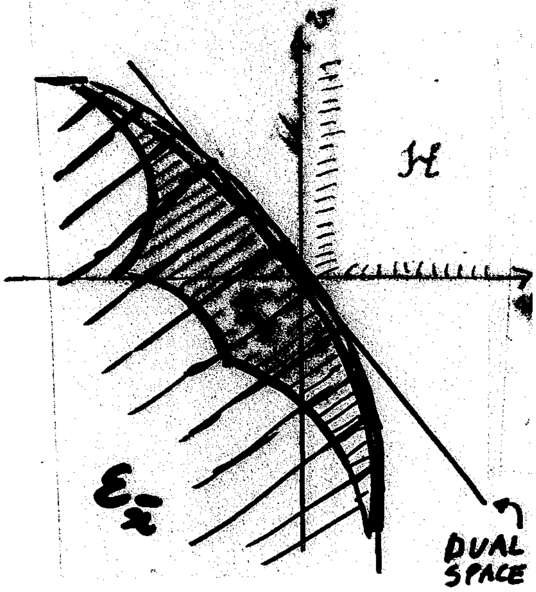
$A_{\bar{x}}(x) = (f(\bar{x})-f(x), g(x))$

$\mathcal{K}_{\bar{x}} = A_{\bar{x}}(X)$

$\mathcal{E}_{\bar{x}} := \mathcal{K} - cl\,\mathcal{H}$

$w : IS\times\Omega\to\mathbb{R}$ s.t. $lev_{>0}\,w \supseteq \mathcal{H}$

WEAK SEPARATION FUNCTION

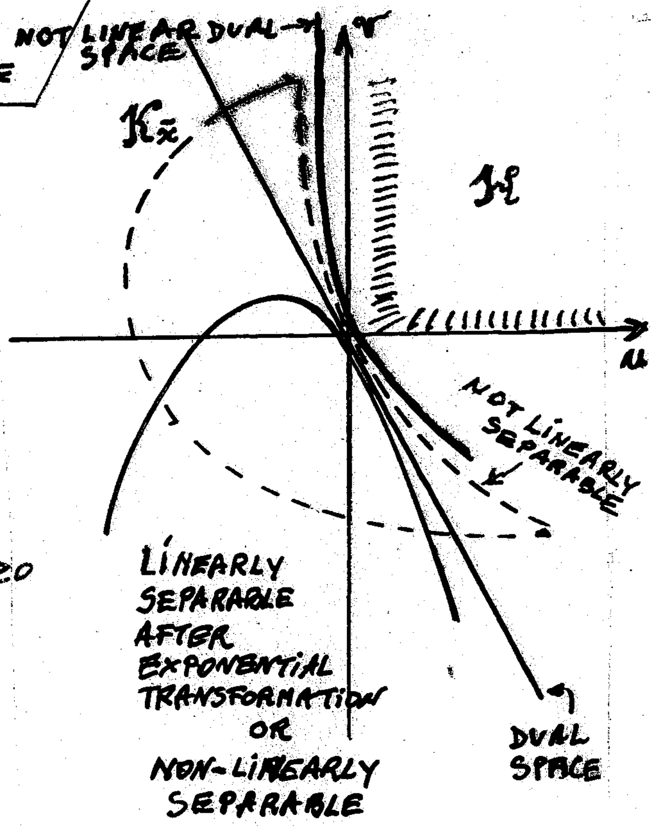


$w = \theta u + \langle\lambda, v\rangle$

LINEAR CASE

$w = \theta u + \sum_i \lambda_i v_i e^{-t_i v_i}$

EXPONENTIAL CASE



$$f(\bar{x}) \leq \overbrace{f(x) - \langle\lambda, g(x)\rangle}^{L(x;\lambda)}\,,\ \forall x\in X$$

$$L(\bar{x};\lambda)\leq L(\bar{x};\bar{\lambda})\leq L(x;\bar{\lambda}),\ \forall x\in X, \forall\lambda\geq 0$$

SADDLE POINT CONDITION

$$\begin{cases} \min \ f(x) = x_1 \\ g(x_1, x_2) = \dfrac{1}{1-x_1^2} + x_2^2 \geq 0 \\ x \in X = \,]-1, 1[\, \times \mathbb{R} \end{cases}$$

$\mathcal{K}$ CLOSED

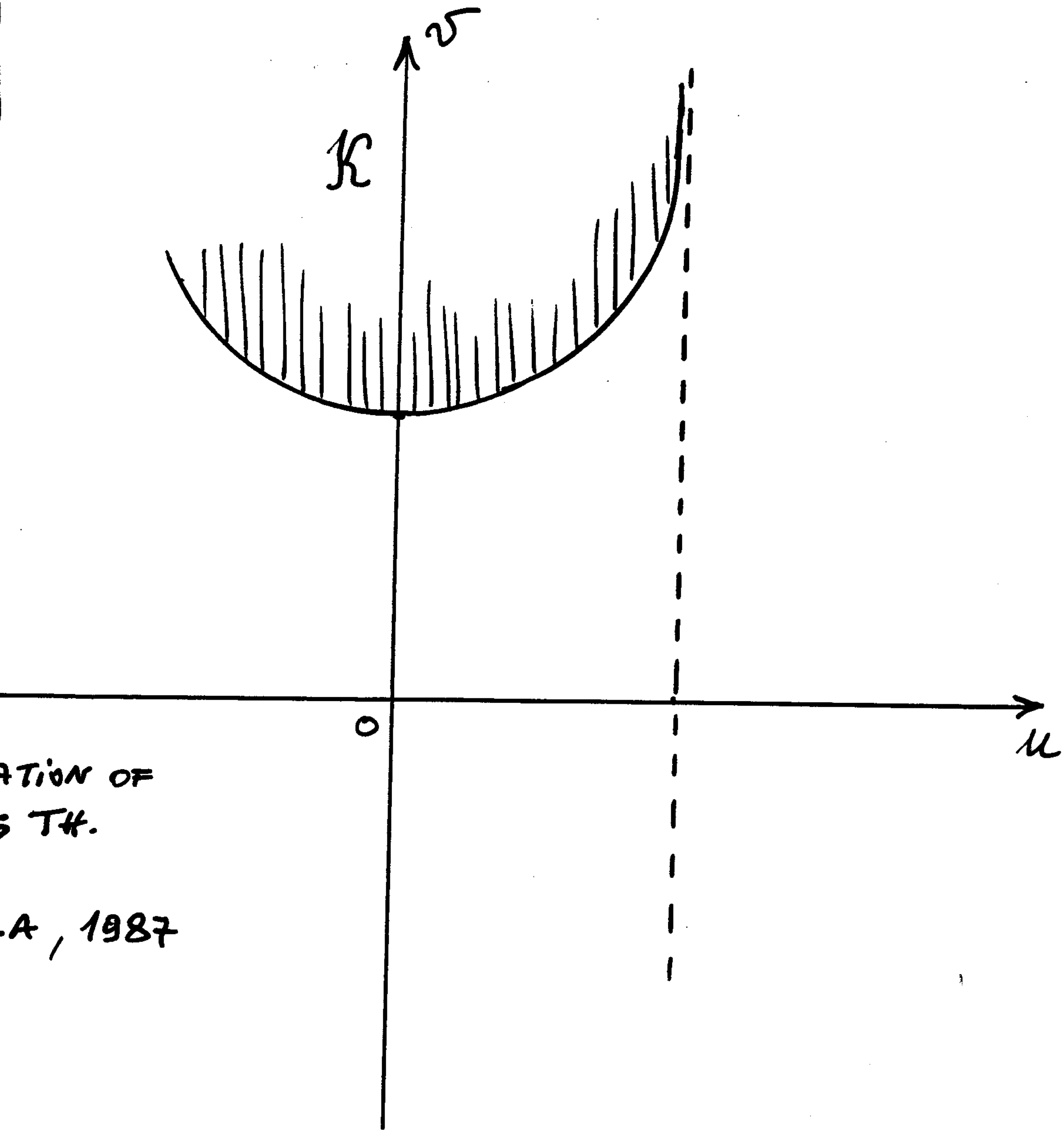
$\mathcal{E}$ NOT CLOSED

$\bar{x} = (0,0)$

$$\mathcal{K} = \left\{ (u, v) \in \mathbb{R} \times \mathbb{R} : \ -1 < u < 1 \ ; \ v \geq \frac{1}{1-u^2} \right\}$$

$$\mathcal{E}(\mathcal{K}) = \left\{ (u, v) \in \mathbb{R} \times \mathbb{R} : \ u < 1 \right\}$$

$\boxed{\begin{array}{l} \exists \inf > -\infty \\ \nexists \min \end{array}}$



GENERALIZATION OF WEIERSTRASS TH.

F. TARDELLA, 1987

$$\begin{cases} \min\ f(x) = x_1 \\ g(x_1, x_2) = \dfrac{1}{1-x_1^2} + x_2^2 \geq 0 \\ x \in X = \,]-1, 1[\, \times \mathbb{R} \end{cases}$$

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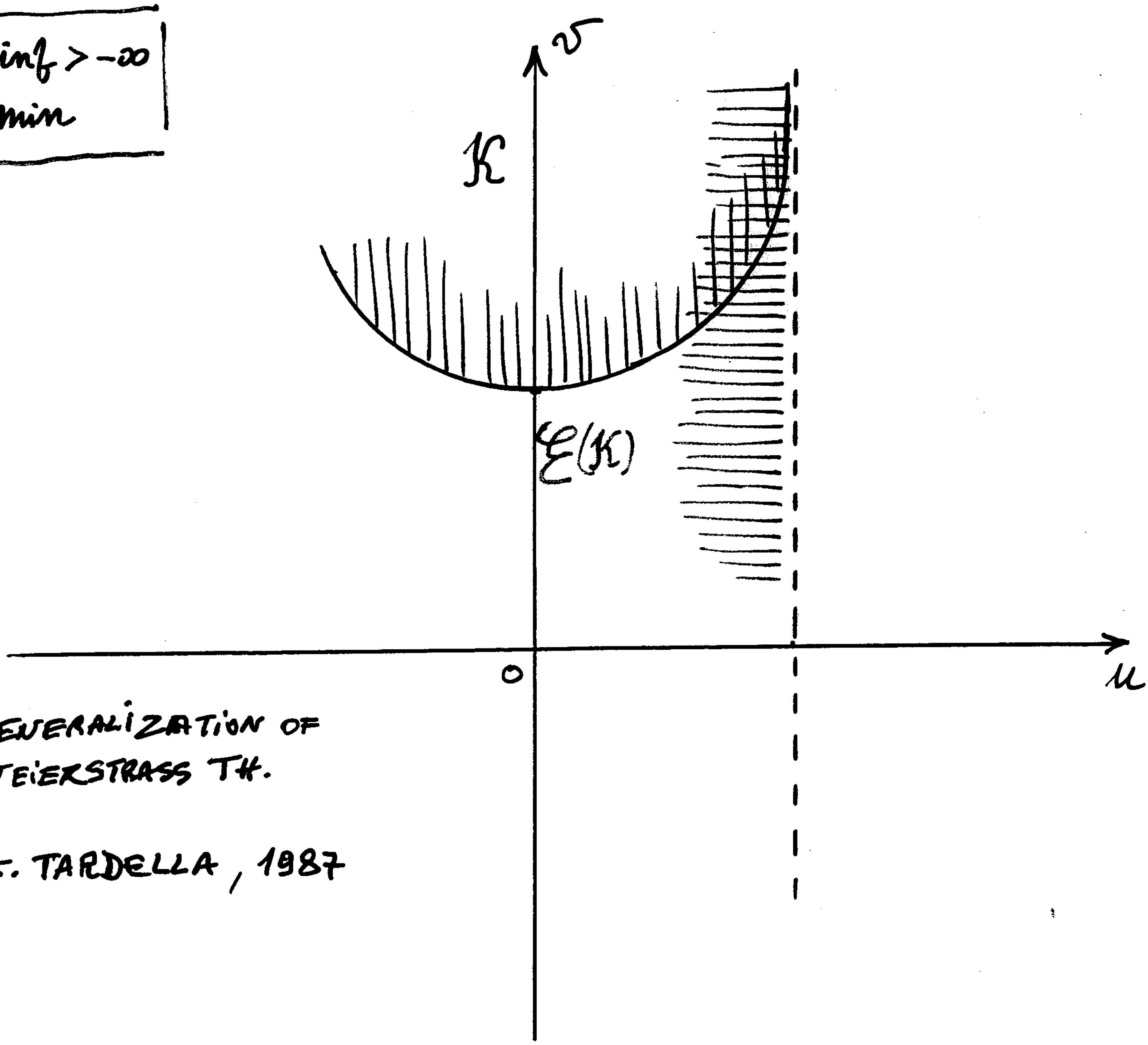
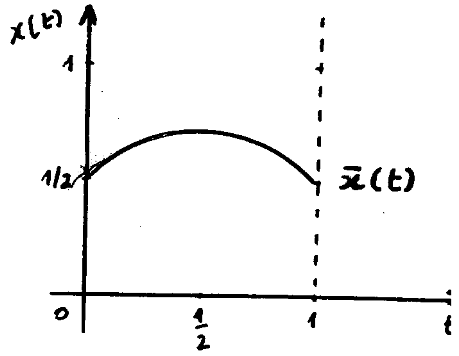
$\boxed{\begin{array}{l} \exists \inf > -\infty \\ \nexists \min \end{array}}$



GENERALIZATION OF WEIERSTRASS TH.

F. TARDELLA, 1987

$$\begin{cases} \min\left[\varphi(x(t)) = \int_0^1 \sqrt{1+x'(t)^2}\; dt\right] \\ g(x(t)) = \int_0^1 x(t)\, dt - \frac{2+\pi}{8} \geq 0 \\ x(t) \in X = \left\{ x(t) \in \mathcal{C}[0,1] \;:\; x(0)=x(1)=1/2 \right\} \end{cases}$$



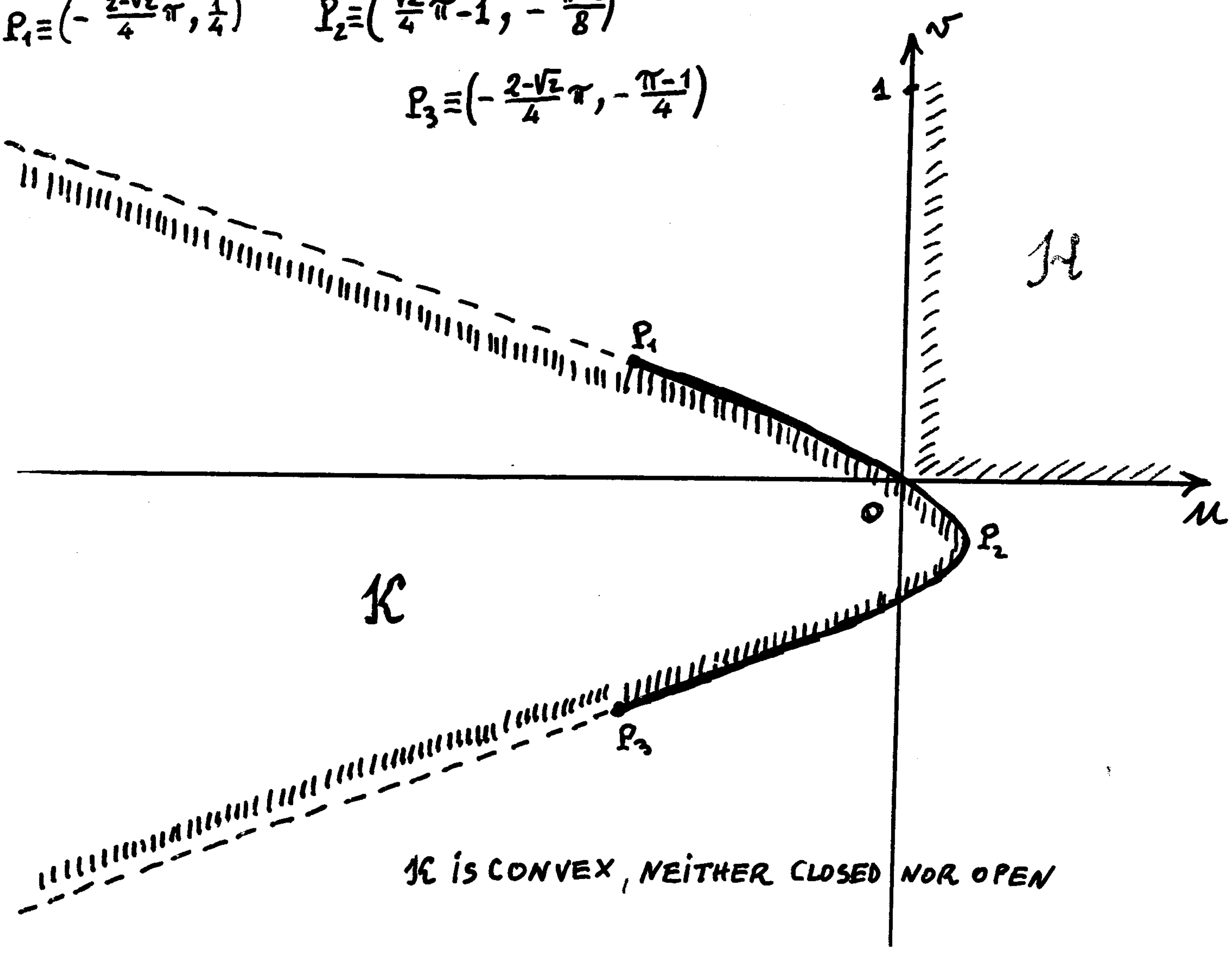
$$\begin{cases} f(x(t)) = \varphi(\bar{x}) - \varphi(x) = \frac{\sqrt{2}}{4}\pi - \int_0^1 \sqrt{1+x'(t)}\; dt > 0 \\ g(x(t)) = \int_0^1 x(t)\, dt - \frac{2+\pi}{8} \geq 0 \\ x(t) \in X \end{cases}$$

$$\bar{x}(t) = \sqrt{-t^2+t+1/4}$$

$\mathcal{H} = \{(u,v) : u>0 \;;\; v \geq 0\}$ $\qquad \mathcal{K} = \{(u,v) : u = f(x(t)) \;;\; v = g(x(t)) \;;\; x \in X\}$

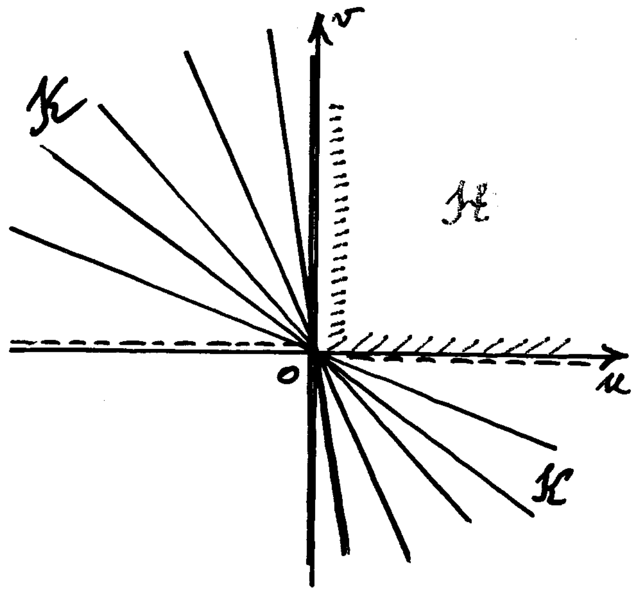
$P_1 \equiv \left(-\frac{2-\sqrt{2}}{4}\pi, \frac{1}{4}\right) \qquad P_2 \equiv \left(\frac{\sqrt{2}}{4}\pi - 1, -\frac{\pi-2}{8}\right)$

$P_3 \equiv \left(-\frac{2-\sqrt{2}}{4}\pi, -\frac{\pi-1}{4}\right)$



𝒦 IS CONVEX, NEITHER CLOSED NOR OPEN

$$\begin{cases} \min \ f(x) = x_1^2 x_2 \\ g(x) = x_2 \geq 0 \\ x = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$



$\bar{x} = (0,0)$ è punto di minimo

$\Updownarrow$

$$\begin{cases} u = f(\bar{x}) - f(x) = -x_1^2 x_2 > 0 \\ v = g(x) = x_2 \geq 0 \\ \quad x \in \mathbb{R}^2 \end{cases}$$ È IMPOSSIBILE

$\Updownarrow$

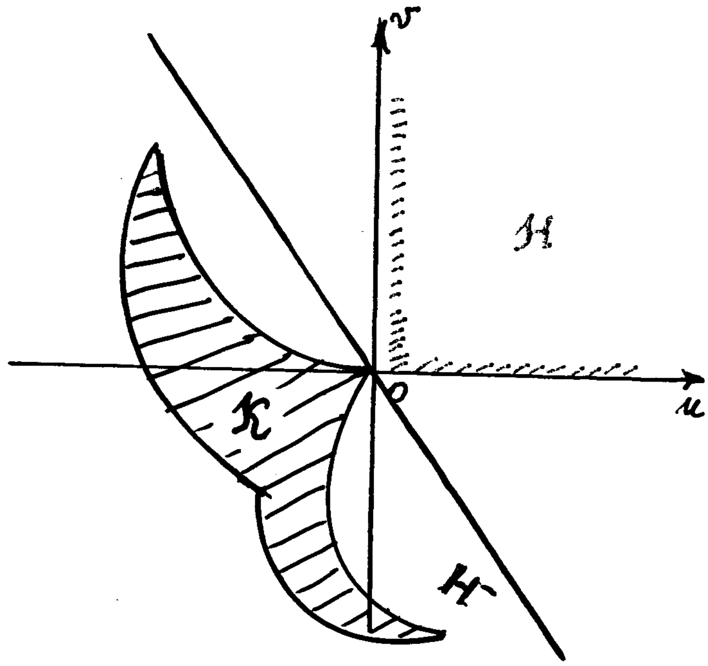
(VERA) $\mathcal{H} \cap \mathcal{K} = \emptyset$

$\Uparrow$

(FALSA) $\mathcal{K} \subseteq \mathcal{H}^-$

$\mathcal{K} = \{(u,v) \in \mathbb{R}^2 : u = -x_1 v, \ x_1 \in \mathbb{R}\}$

$$\begin{cases} \min \ [x_1 + 2x_2 + 3x_1(1-x_1) + 3x_2(1-x_2)] \\ 2x_1 + x_2 - 2 \geq 0 \\ (x_1, x_2) \in X = [0,1] \times [0,1] \end{cases}$$



$\bar{x} = (1,0)$ è punto di minimo

$\Updownarrow$

$1 - f(x) > 0 \quad g(x) \geq 0$ È IMPOSS.

$\Updownarrow$

(VERA) $\mathcal{H} \cap \mathcal{K} = \emptyset$

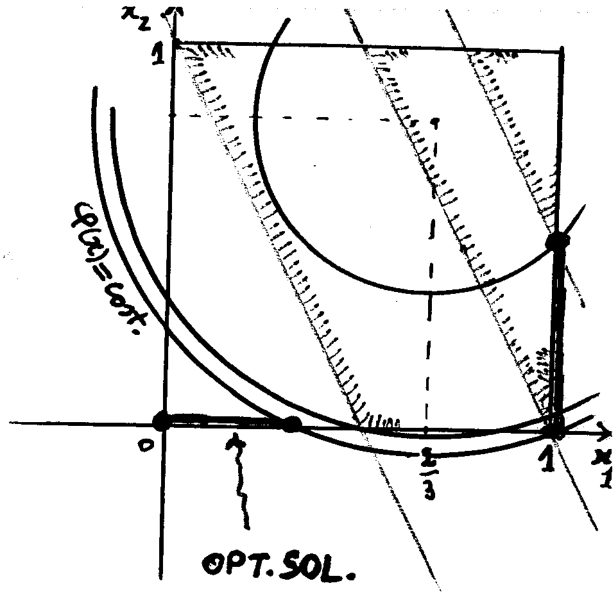
$\Uparrow$

(VERA) $\mathcal{K} \subseteq \mathcal{H}^-$

$\mathcal{K} = \{(u,v) \in \mathbb{R}^2 : 3v^2 - (12x_1 - 7)v + 15x_1^2 - 18x_1 + 3, \ x_1 \in \mathbb{R}\}$

$$P: \begin{cases} \min\left[\varphi(x) = x_1 + 2x_2 + 3x_1(1-x_1) + 3x_2(1-x_2)\right] \\ g(x;\xi) = 2x_1 + x_2 - \xi \geq 0 \\ x \in X = \{x \in \mathbb{R}^2 : 0 \leq x_i \leq 1,\ i = 1,2\} \end{cases}$$

$$0 \leq \xi \leq 3$$



$$\begin{cases} f(x;\bar{x}) = \varphi(\bar{x}) - \varphi(x) > 0 \\ g(x;\xi) \geq 0 \qquad x \in X \end{cases}$$

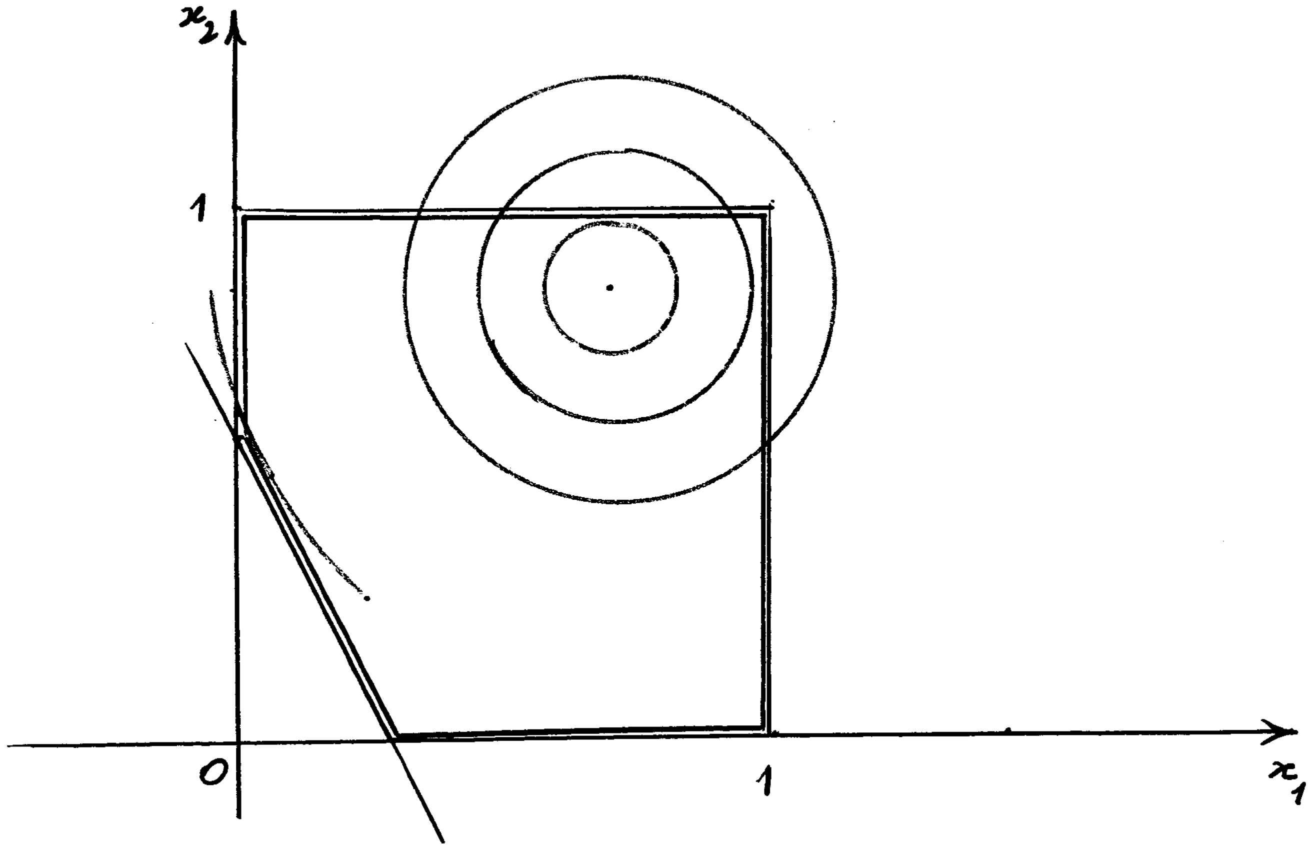
$$\mathcal{H} = \{(u,v) : u > 0\ ;\ v \geq 0\}$$

$$\mathcal{K}(\bar{x};\xi) = \{(u,v) \in \mathbb{R}^2 : u = f(x;\bar{x})\ ;\ v = g(x;\xi)\ ;\ x \in X\}$$

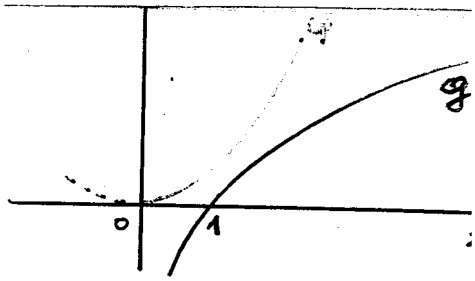
$$\mathcal{K}: \begin{cases} u = 3v^2 - (12x_1 + 5 - 6\xi)v + 15x_1^2 + (6 - 12\xi)x_1 + 3\xi^2 - 5\xi + \varphi(\bar{x}) \\ 2x_1 - \xi \leq v \leq 2x_1 - \xi + 1 \qquad 0 \leq x_1 \leq 1 \end{cases}$$

P is A CONTINUOUS FORMULATION OF AN INTEGER PROGRAM

$$\begin{cases} \min \quad f(x_1, x_2) = x_1 + 2x_2 + 3x_1(1-x_1) + 3x_2(1-x_2) \\ g(x;\xi) = 2x_1 + x_2 - \xi \geq 0 \qquad\qquad (0 \leq \xi \leq 3) \\ x \in X = \{x \in \mathbb{R}^2 : \ 0 \leq x_1 \leq 1 \ , \ 0 \leq x_2 \leq 1\} \end{cases}$$



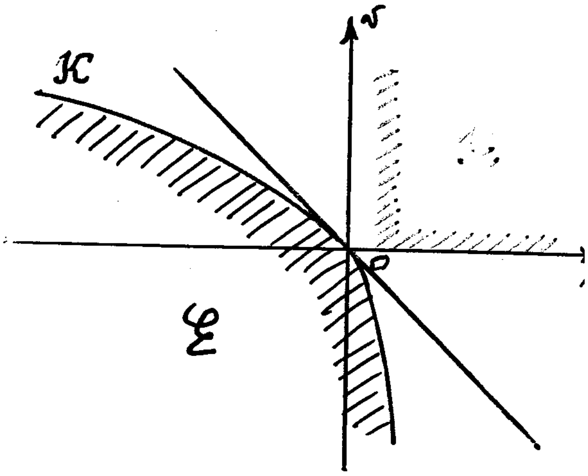
$$\begin{cases} \min\ [\varphi(x) = x^2] \\ g(x) = \log x \geq 0 \qquad \bar{x} = 1 \\ x \in X =]0, +\infty[\end{cases}$$



$$f(x) = \varphi(\bar{x}) - \varphi(x) = 1 - x^2$$

$$g(x) = \log x$$

$$\mathcal{K} = \{(u, v) :\ u = 1 - e^{2v}\}$$

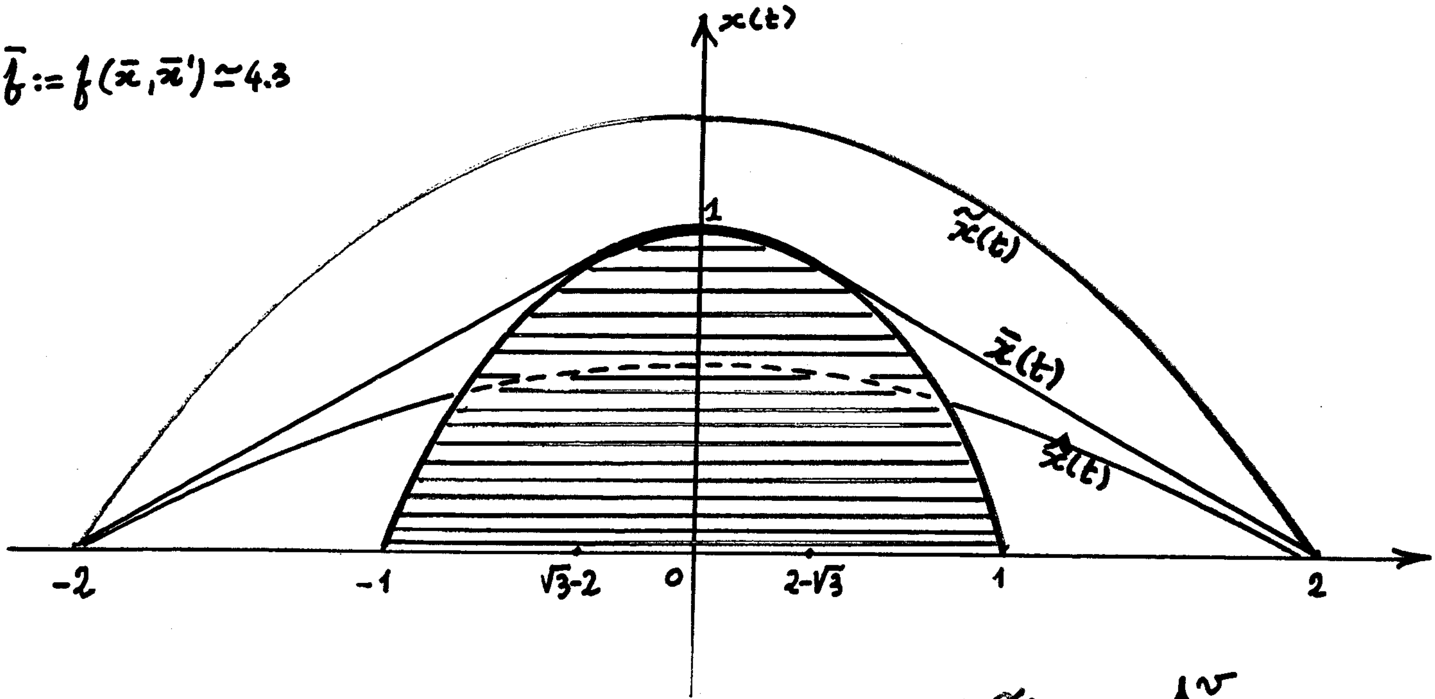


φ, g CONTINUOUS, BUT X NOT COMPACT
HENCE WEIERSTRASS T. IS NOT APPL.
HOWEWER $\mathcal{K}$ IS (cl $\mathcal{H}$)-COMPACT
AND $\mathcal{E}$ IS CLOSED, AND HENCE MINIMUM $\exists$

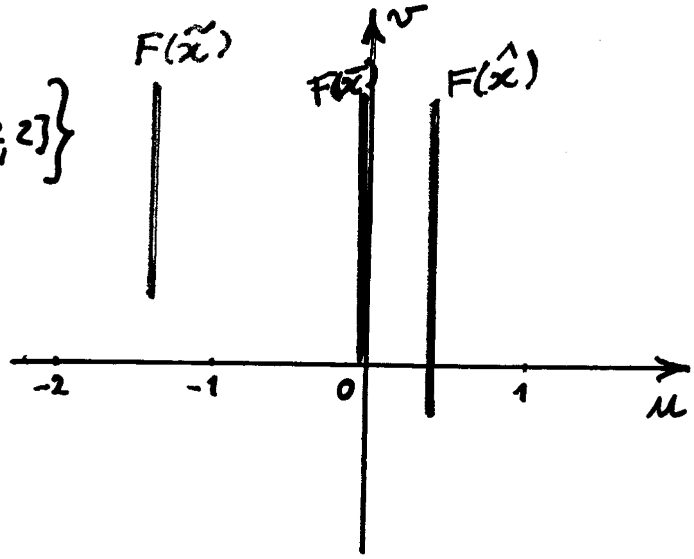
MIN. $\exists$ IF $\mathcal{E} \cap \{(u, v) : v = 0\}$ IS CLOSED

$$\begin{cases} \min f(x,x') = \displaystyle\int_{-2}^{2} \sqrt{1+x'^2}\,dt \\ g(t,x,x') = t^2-1+x(t) \geq 0 \\ x \in X = \{x \in C^0[-2,2] : x(-2) = x(2) = 0\} \end{cases}$$

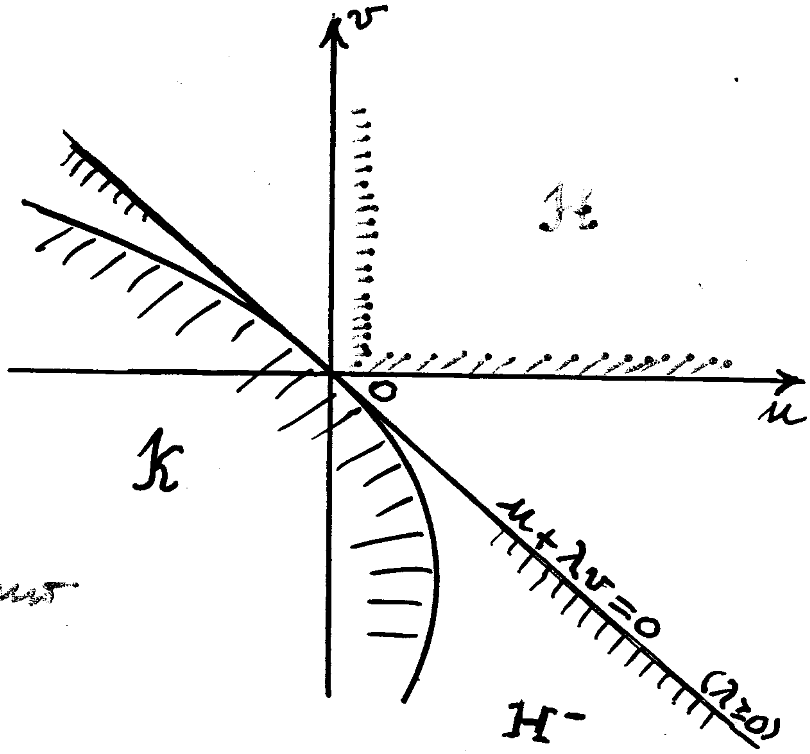
$\bar{f} := f(\bar{x},\bar{x}') \simeq 4.3$



$$F(x) := \left\{ \left(\bar{f} - f(x,x'),\ g(t,x,x')\right) : t \in [-2,2] \right\}$$



$$\begin{cases} \min f(x) \\ g(x) \geq 0 \\ x \in X \end{cases} \Big\} R$$



$\bar{x} \in R$ è punto di minimo

$$\Updownarrow$$

$$\begin{cases} u = f(\bar{x}) - f(x) > 0 \\ v = g(x) \geq 0 \\ x \in X \end{cases}$$ È IMPOSSIBILE

$$\Updownarrow$$

$$\mathcal{H} \cap \mathcal{K} = \emptyset$$

$$\Uparrow$$

$$\mathcal{K} \subseteq H^-$$

$$\Updownarrow$$

$\exists \lambda: \quad f(\bar{x}) - f(x) + \lambda g(x) \leq 0, \ \forall x \in X$

$\exists \lambda: \quad f(\bar{x}) \leq f(x) - \lambda g(x), \quad \forall x \in X$

$\exists \lambda: \quad f(\bar{x}) \leq L(x;\lambda), \quad \forall x \in X$

$f(\bar{x}) \leq \sup_{\lambda} \inf_{x \in X} L(x;\lambda)$ DUALE LAGRANGIANO

$$\boxed{\begin{aligned} \mathcal{H} &= \{(u,v): u > 0, \ v \geq 0\} \\ \mathcal{K} &= \{(u,v): u = f(\bar{x}) - f(x), \ v = g(x), \ x \in X\} \end{aligned}}$$

$$\boxed{L(x;\lambda) = f(x) - \lambda g(x)}$$
LAGRANGIANA

λ = MOLTIPLICATORE DI LAGRANGE

HOMOGENEIZATION AND SEMISTATIONARITY

$$\begin{cases} \min f(x) \\ g(x) \geq 0 \\ x \in X \end{cases}$$

H HILBERT SPACE
$X \subseteq H$
$f : X \to \mathbb{R}$
$g : X \to \mathbb{R}^m$

$I = \{1 \dots m\}$
$g(x) = (g_1(x), \dots, g_m(x))$
$N_\rho(x)$ NEIGHBOURHOOD

PARTICULAR CASE $\quad f(x) = \int_a^b \psi_0(t, x(t), x'(t))\,dt \quad g_i(x) = \int_a^b \psi_i(t, x(t), x'(t))\,dt$

$X = C^1[a,b]$

LEMMA 1 (HOMOGENEIZATION)

$f, -g_i$ φ-differentiable at $\bar{x} \in X$. If $\bar{x}$ is a m.p. then SYSTEM

$$(*) \begin{cases} D_\varphi f(\bar{x}; z) < 0 \\ D_\varphi g_i(\bar{x}; z) > 0 \quad, i \in I^0 = \{i \in I : g_i(\bar{x}) = 0 \,;\, \varepsilon_i(\bar{x}; z) \not\equiv 0\} \\ g_i(\bar{x}) + D_\varphi g_i(\bar{x}; z) \geq 0 \quad, i \in I \setminus I^0 \end{cases}$$

is IMPOSSIBLE

$$\mathcal{H}_h = \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0 \,;\, v_i > 0 \ i \in I^0 \,;\, v_i \geq 0 \ i \in I \setminus I^0\}$$

$$\mathcal{K}_h = \{ \quad \text{"} \quad : u = -D_\varphi f(\bar{x}; z) \,;\, v_i = g_i(\bar{x}) + D_\varphi g_i(\bar{x}; z) \,; \ i \in I^0 ; z \in Z\}$$

$(*)$ IMPOSSIBLE $\iff \mathcal{H}_h \cap \mathcal{K}_h = \emptyset$

IF f, g DIFFERENTIABLE, THEN LEMMA 1 BECOMES LINEARIZ. LEMMA OF ABADIE (1967) AND $(*)$ BECOMES:

$$\langle f'(\bar{x}), z \rangle < 0 \quad ; \quad \langle g_i'(\bar{x}), z \rangle > 0 \ i \in I^0 \,; \ \langle g_i'(\bar{x}), z \rangle \geq 0 \,, \ i \in I \setminus I^0$$

$\bar{x} \in X$ LOWER SEMISTATIONARY POINT OF $\min_{x \in X} f(x)$ IFF

$$(*) \qquad \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0$$

PROPOSITION

- $\bar{x}$ local m.p. of f on $X \Rightarrow \bar{x}$ l.s.p.
- if f, X convex $\Rightarrow$ l.s.p. is a global m.p.
- if f is DIFFERENTIABLE $\Rightarrow$ $(*)$ BECOMES $\langle f'(\bar{x}), x - \bar{x} \rangle \geq 0, \forall x \in X$ OR $f'(\bar{x}) = 0$ if $\bar{x} \in \operatorname{int} X$
- IF f, X CONVEX $\Rightarrow$ $(*)$ BECOMES $f'(\bar{x}; x - \bar{x}) \geq 0, \forall x \in X$

$L(x; \theta, \lambda) = \theta f(x) - \langle \lambda, g(x) \rangle \qquad L(x;\theta,\lambda) = \int_a^b \left[\theta \psi_0(t,x,\dot{x}) - \sum_{i=1}^m \lambda_i \psi_i(t,x,\dot{x})\right] dt$

LEMMA 2 (SEMISTATIONARITY)

$f, -g$ G-differentiable with $G \subseteq \mathcal{G}$

(i) $-(\theta, \lambda) \in (\mathcal{K}_h - \bar{k})^* \Rightarrow \bar{x}$ l.s.p. of L

In $(*)$ liminf collapses to lim if $\lim_{z \to 0} \mathcal{D}_\varphi f(\bar{x}; \frac{z}{\|z\|})$ and $\lim_{z \to 0} \mathcal{D}_\varphi g_i(\bar{x}; \frac{z}{\|z\|})$ exist

(ii) f, g differentiable $(G = \mathcal{L})$ then (i) becomes

$$-(\theta, \lambda) \in (\mathcal{K}_h - \bar{k})^\perp \Rightarrow L'_x(\bar{x}; \theta, \lambda) = 0$$

$$\mathcal{E}(S) = S - \mathrm{cl}\,\mathcal{H}$$ CONIC EXTENSION OF S WITH RESPECT TO CONE $\mathrm{cl}\,\mathcal{H}$

LEMMA 3 (CONIC EXTENSION)

(i) $\mathcal{H} \cap \mathcal{K} = \emptyset \iff \mathcal{H} \cap \mathcal{E}(\mathcal{K}) = \emptyset$

(ii) $\mathcal{H}_h \cap \mathcal{K}_h = \emptyset \iff \mathcal{H}_h \cap \mathcal{E}(\mathcal{K}_h) = \emptyset$

(iii) $f, -g$ $\mathcal{C}$-DIFFER. $\Rightarrow \mathcal{E}(\mathcal{K}_h)$ CONVEX

(iv) f, g DIFFERENZ. $\Rightarrow \mathcal{E}(\mathcal{K}_h)$ is SUM OF AFFINE VARIETY AND $\mathbb{R}_-^{1+2}$

LEMMA 4 (SIGN OF MULTIPLIERS)

$f, -g$ $\mathcal{C}$-differentiable at $\bar{x} \in X$:

$$\bar{x} \text{ m.p.} \implies \mathbb{K} := (\mathcal{K}_h - \bar{k})^* \cap [\mathbb{R}_-^{1+m} \setminus \{0\}] \neq \emptyset$$

f, g DIFFER. at $\bar{x}$:

$$\bar{x} \text{ m.p.} \implies (\mathcal{K}_h - \bar{k})^\perp \cap [\quad " \quad] \neq \emptyset$$

LEMMA 5 (COMPLEMENTARITY)

$f, -g$ $\mathcal{C}$-differ. at $\bar{x} \in X$:

$$\bar{x} \text{ m.p.} \implies \exists (-\theta, -\lambda) \in \mathbb{K} \text{ s.t. } \langle \bar{\lambda}, g(\bar{x}) \rangle = 0$$

LEMMA 6 (SIGN OF DIRECTIONAL DERIVATIVE)

(i) $f, -g$ G-DIFFERENTIABLE:

$$\bar{x} \text{ l.s.p. OF } L \; ; \theta \geq 0 ; \lambda \geq 0 \implies \inf_{z \in S} \mathcal{D}_G L(\bar{x}; z; \theta, \lambda) \geq 0$$

ii) f, g DIFFERENTIABLE:

$$" \implies \mathcal{V}_L(\bar{x}; z; \theta, \lambda) = 0$$

iii) f, g DIFFER. AND $X = H = \mathbb{R}^n$:

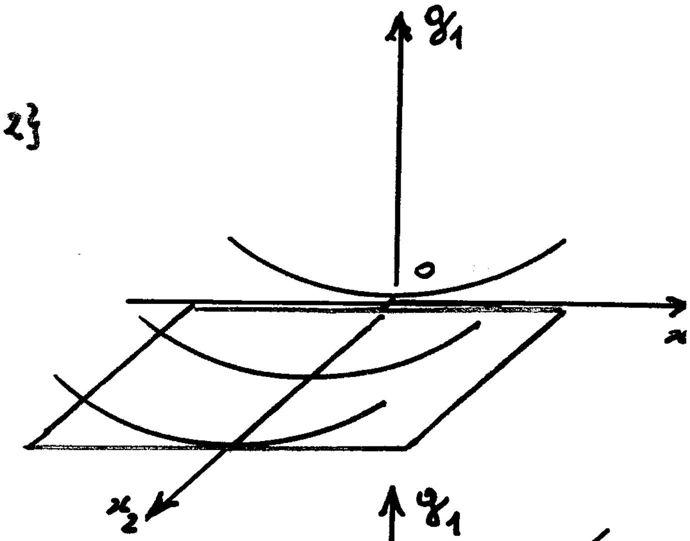
$$" \implies \langle L'_x(\bar{x}; \theta, \lambda), z \rangle = 0, \forall z \in S$$

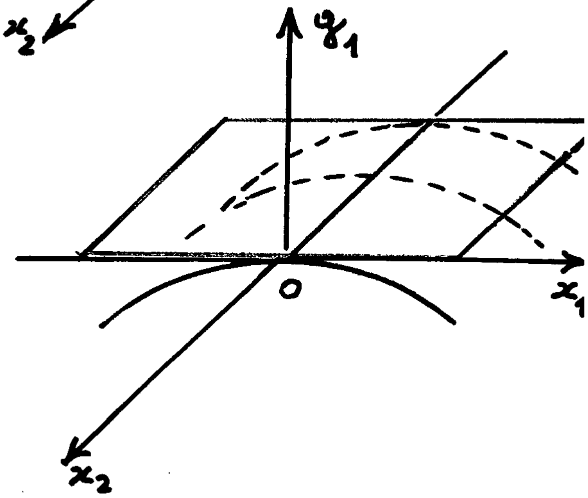
EXAMPLE

$\bar{x}=(0,0) \quad \mathcal{J}=\{1,2\}$

$\mathcal{J}^0=\{1\} \quad \mathcal{J}^+=\{2\}$

$$\begin{cases} \min f(x)=x_1 \\ g_1(x)=\begin{Bmatrix} -x_2^2-x_1^2 \\ x_2^2+x_1^2 \end{Bmatrix}=0 \\ g_2(x)\equiv 1 \geq 0 \qquad x\in\mathbb{R}^2 \end{cases}$$





$$\mathcal{K}=\left\{(u,v_1,v_2): v_2=1\,;\, v_1=\begin{Bmatrix} -x_2^2-u^2, \text{ if } x_2\leq 0 \\ x_2^2+u^2, \text{ if } x_2>0 \end{Bmatrix}\right\}$$

$$\mathcal{K}_h=\{ \quad '' \quad : u=-z_1\,;\, v_1=0\,;\, v_2=1\}$$

$$L'_x=\theta f'(\bar{x})-\lambda_1 g_1'(\bar{x})-\lambda_2 g_2'(\bar{x})=$$

$$=\begin{pmatrix} \theta\pm 2\lambda_1\bar{x}_1 \\ \pm 2\lambda_1\bar{x}_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

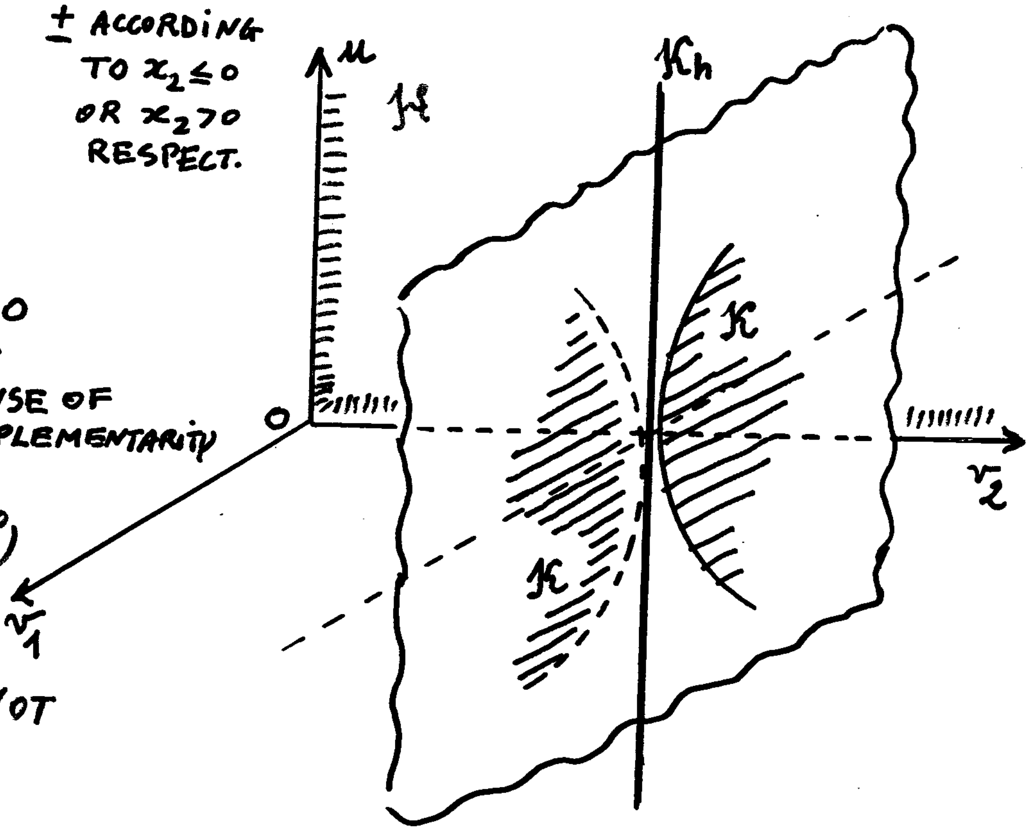
$\pm$ ACCORDING TO $x_2\leq 0$ OR $x_2>0$ RESPECT.

$$\Updownarrow$$

$$\theta=0 \quad \lambda_1>0 \quad \lambda_2=0$$

$\uparrow$ BECAUSE OF COMPLEMENTARITY

THESIS OF CLASSIC THEOREM ($g_i\in C^1, i\in\mathcal{J}^0$) HOLDS, BUT ASSUMPTION DOES NOT



THEOREM

$f, -g$ $\mathcal{C}$-differentiable at $\bar{x} \in X$. If $\bar{x}$ is a m.p. then $\exists\, \bar{\theta} \in \mathbb{R}$ $\exists\, \bar{\lambda} \in \mathbb{R}^m$ s.t.

(*) $\inf_{z \in S} \mathcal{D}_{\varphi} L(\bar{x}; z; \bar{\theta}, \bar{\lambda}) \geq 0 \qquad \langle \bar{\lambda}, g(\bar{x}) \rangle = 0$

$$g(\bar{x}) \geq 0 \qquad \bar{\theta} \geq 0 \qquad \bar{\lambda} \geq 0 \qquad (\bar{\theta}, \bar{\lambda}) \neq 0$$

or

$$0 \in \bar{\theta}\, \partial_{\varphi} f(\bar{x}) - \sum_{i \in I} \lambda_i\, \partial_{\varphi} [-g_i(\bar{x})]$$

∂_{φ} becomes ∂ if $X, f, -g$ are convex

IF f, g are differentiable, (*) becomes

(**) $\qquad \nabla_L = 0 \quad$ along $x = \bar{x}$

In the "ISOPERIMETRIC CASE" (**) becomes

(EULER EQ.) $\Psi'_x(t, \bar{x}, \bar{x}'; \bar{\theta}, \bar{\lambda}) - \frac{d}{dt} \Psi'_{x'}(t, \bar{x}, \bar{x}'; \bar{\theta}, \bar{\lambda}) = 0$

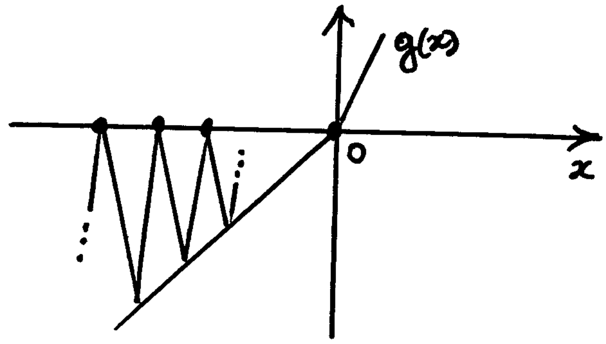
Ψ = integrand of L

If f, g are diff., $X = \Xi = \mathbb{R}^n$ then (*) becomes

$$L'_x(\bar{x}; \bar{\theta}, \bar{\lambda}) = 0$$

WITH THE ORDINARY CONCEPT OF SEMIDERIVATIVE AND CORRESPONDING SUBDIFFERENTIAL WE CANNOT ESTABLISH NEC. COND.S, LIKE THE ABOVE, FOR LOC. LIP. FUNCT.S. FOR INSTANCE

$\min f(x)$ s.t. $g(x) \geq 0$



A NEC. COND. CAN BE ACHIEVED, IF THE CONCEPT OF SEMIDERIVATIVE IS STRENGTHENED AS F.H. CLARKE. THEN WE OBTAIN (HIRIART-URRUTY, Appl. Math. Optim., 1979):

FUNCT.S LOC. LIP. IF $\bar{x}$ IS A MIN. POINT OF A MIXED PROBL. THEN $\exists \lambda_1 \dots \lambda_p$ $\exists \lambda_{p+1} \geq 0 \dots \lambda_m \geq 0$ S.T.

$$0 \in \partial_C \left(\theta\varphi - \sum_i \lambda_i g_i\right) \qquad \lambda_i g_i(\bar{x}) = 0, \; i = p+1 \dots m$$

CLARKE SUBDIFF.

LAGRANGE AND PONTRYAGIN MAX PRINCIPLES

$$\begin{cases} \min \quad x_0 := \int_{t_0}^{t_1} f_0(x(t), \xi(t))\, dt & \left(\frac{\partial f_i}{\partial x_j} \in C^0[t_0, t_1] \right) \\ \frac{dx_i}{dt} = f_i(x, \xi) \quad , i = 1 \dots m \; ; \; x(t_0) = x^0 \; ; \; x(t_1) = x^1 \\ \xi \in U \qquad x \in X = C^1[t_0, t_1] \end{cases}$$

..MAX PRINC. $\bar{x}$ min. point $\Rightarrow \exists\, \theta, \lambda, \mu$, with $(\theta, \lambda) \underset{\neq}{\geq} 0$, S.T. the problem

(*) $\quad \sup \; \theta u + \langle \lambda, v^+ \rangle + \langle \mu, v^0 \rangle \quad , (u, v^+, v^0) \in \mathcal{K}, \; v^+ \geq 0, \; v^0 = 0,$

has the same extremum as the given problem. (*) is equivalent to:

$$\inf \; L(x; \theta, \lambda, \mu) \quad , \; x \in R,$$

which becomes:

(**) $\quad \inf \int_{t_0}^{t_1} \left\{ \theta f_0 - \sum_1^m \lambda_i \omega_i(t) \left[\frac{dx_i}{dt} - f_i(x, \xi) \right] \right\} dt \quad , (x, \xi) \in R$

Set $\psi_0 = -\theta \leq 0$, $\psi_i = -\lambda_i \omega_i(t)$, $\quad H(x, \xi; \psi) = \sum_0^m \psi_i f_i(x, \xi)$.

(**) becomes:

(I) $\quad -\sup \int_{t_0}^{t_1} \left[H(x, \xi; \psi) - \sum_1^m \psi_i \frac{dx_i}{dt} \right] dt \quad , (x, \xi) \in R$

By integrating:

(II) $\quad -\sup \int_{t_0}^{t_1} \left[H(x, \xi; \psi) + \sum_{i=1}^m x_i \frac{d\psi_i}{dt} \right] dt - \sum_{i=1}^m \left[\psi_i x_i \right]_{t_0}^{t_1} \quad , (x, \xi) \in R$

By equating to zero the gradient of the integrand of (I) with respect to ψ, and that of (II) with respect to x, we find:

$$\boxed{\frac{\partial H}{\partial \psi_i} - \frac{dx_i}{dt} = 0} \quad , \quad \boxed{\frac{\partial H}{\partial x_i} + \frac{d\psi_i}{dt} = 0 \quad , i = 1, \dots, m}$$

These equations, together with the

$$\boxed{\underset{\xi}{\mathrm{Max}} \; H(x, \xi; \psi)}$$

are the PONTRYAGIN MAXIMUM PRINCIPLE

$$\min \int_{t_0}^{t_1} dt \qquad \frac{d^2 y(t)}{dt^2} = \xi(t) \qquad -1 \le \xi(t) \le 1 \qquad \text{COND. CONTORNO PER } y, y'$$

SI PONE: $x_1(t) = y(t)$ $\quad x_2(t) = \frac{dy(t)}{dt}$ $\quad$ IL PROBLEMA DIVIENE:

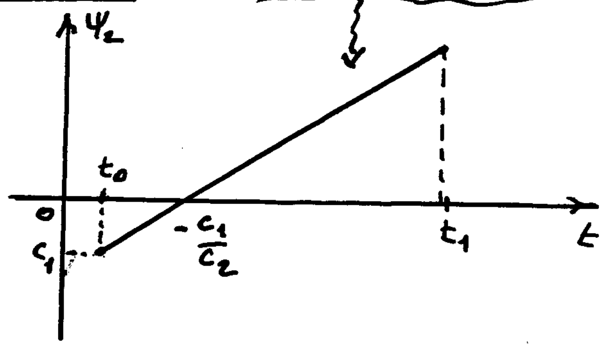
$$\min \int_{t_0}^{t_1} dt \qquad \frac{dx_1(t)}{dt} = x_2(t) \qquad \underbrace{\boxed{\frac{dx_2(t)}{dt} = \xi(t) \quad -1 \le \xi \le 1}}_{\text{ODE}} \qquad \text{COND. CONTORNO}$$

$$H(x_1, x_2, \xi; \psi_0, \psi_1, \psi_2) = \psi_0 + \psi_1 x_2 + \psi_2 \xi$$

LA 2ª CONDIZIONE DI PONTRYAGIN DIVIENE:

$$\begin{cases} \psi_1'(t) = 0 \\ \psi_1(t) + \psi_2'(t) = 0 \end{cases} \qquad \begin{cases} \psi_1(t) = c_1 \\ \psi_2(t) = c_1 + c_2 t \end{cases}$$

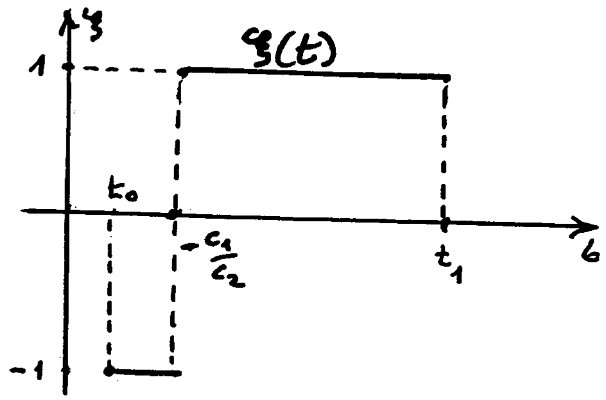
c_1, c_2 DIPENDONO DALLE COND. AL CONT. E LO STESSO PER



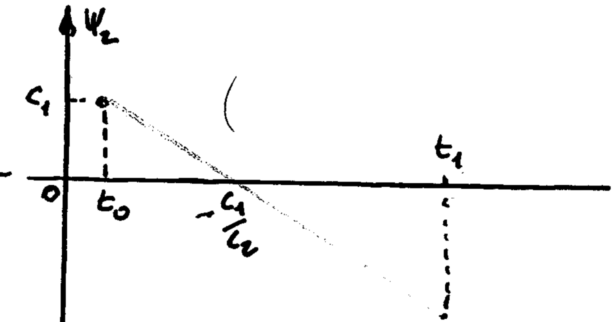
LA 3ª COND. DI PONTRYAGIN DIVIENE:

$$\max_{-1 \le \xi(t) \le 1} [c_1 x_2(t) + (c_1 + c_2 t)\,\xi(t)]$$

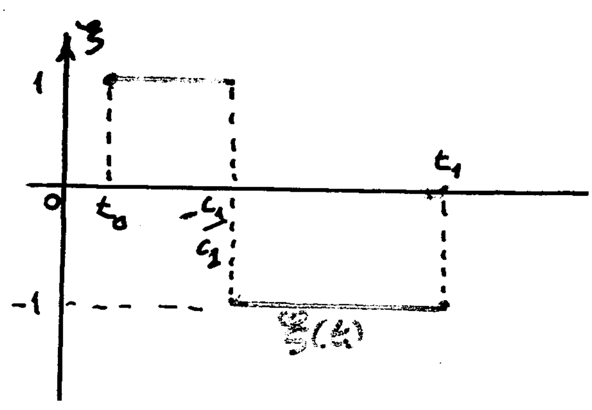
$$\begin{cases} t \in [t_0, -\frac{c_1}{c_2}] \Rightarrow \max = c_1 x_2(t) - c_1 - c_2 t \\ t \in [-\frac{c_1}{c_2}, t_1] \Rightarrow \max = c_1 x_2(t) + c_1 + c_2 t \end{cases}$$



$$t \in [t_0, -\tfrac{c_1}{c_2}] \Rightarrow \max = c_1 x_2(t) + c_1 + c_2 t$$
$$t \in [-\tfrac{c_1}{c_2}, t_1] \Rightarrow \max = c_1 x_2(t) - c_1 - c_2 t$$



ODE REGOLA L'EVOLUZIONE NEL TEMPO DI VARI SISTEMI FISICI. AD ES.: MOTO DI UN SISTEMA MATERIALE DI MASSA UNITARIA, SOLLECITATO DA UNA FORZA $\xi(t)$; IN TAL CASO, x_2 È UNA DELLE COORDINATE DEL BARICENTRO (PUNTO MATERIALE) DEL SISTEMA, E $\xi(t)$ LA COMPONENTE DELLA FORZA APPLICATA, SECONDO LO STESSO ASSE COORDINATO.

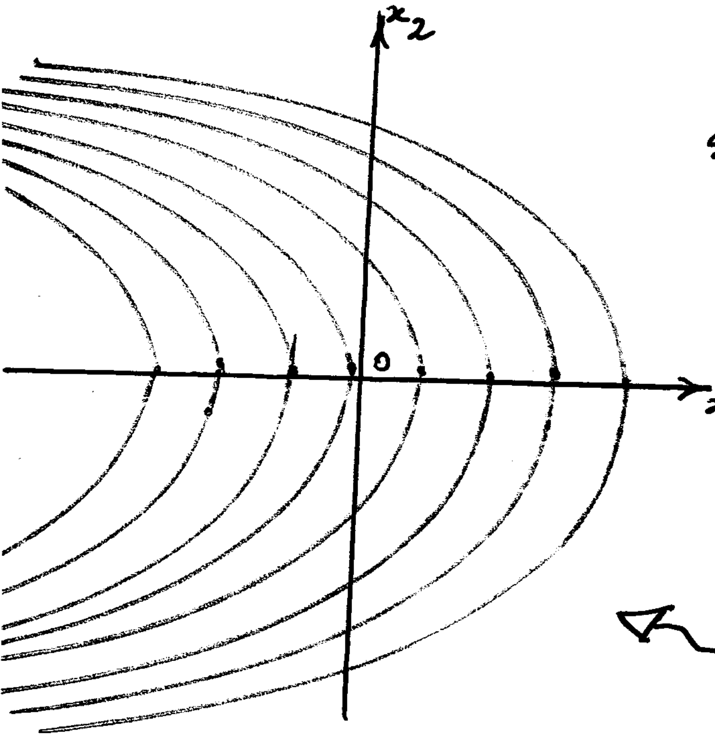


LA 1ª CONDIZIONE DI PONTRYAGIN DÀ LUOGO A:

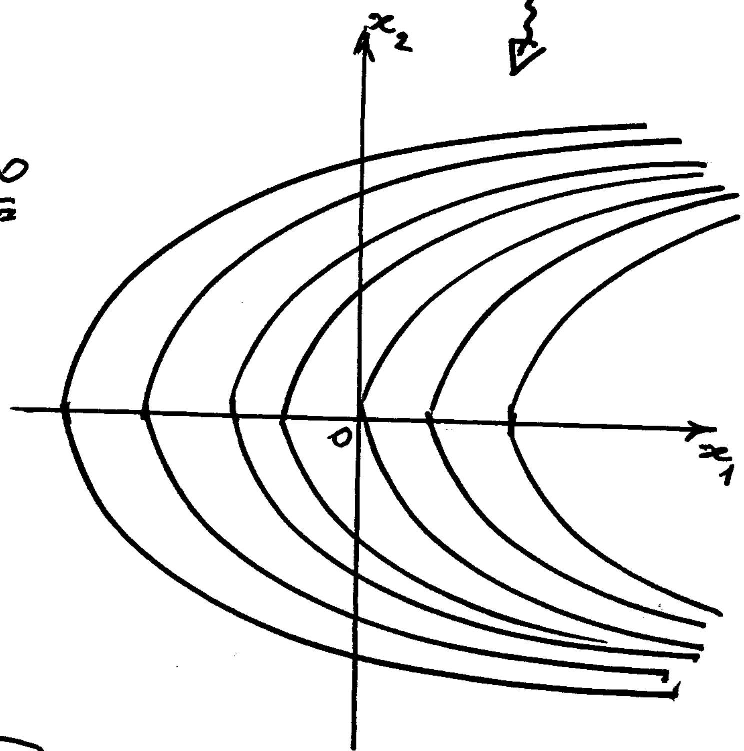
NEL CASO IN CUI IL CONTROLLO È POSTO AL MAX.: $\xi(t) = 1$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = 1 \end{cases} \quad \begin{cases} x_1 = \frac{1}{2}t^2 + c_3 t + c_4 \\ x_2 = t + c_3 \end{cases} \quad \begin{cases} x_1 = \frac{1}{2}x_2^2 \overbrace{- \frac{1}{2}c_3^2 + c_4}^{k} \\ t = x_2 - c_3 \end{cases} \quad \boxed{x_1 = \frac{1}{2}x_2^2 + k}$$

EQUAZIONI ORARIE



SPAZIO DELLE FASI



NEL CASO IN CUI IL CONTROLLO È POSTO AL MIN.: $\xi(t) = -1$

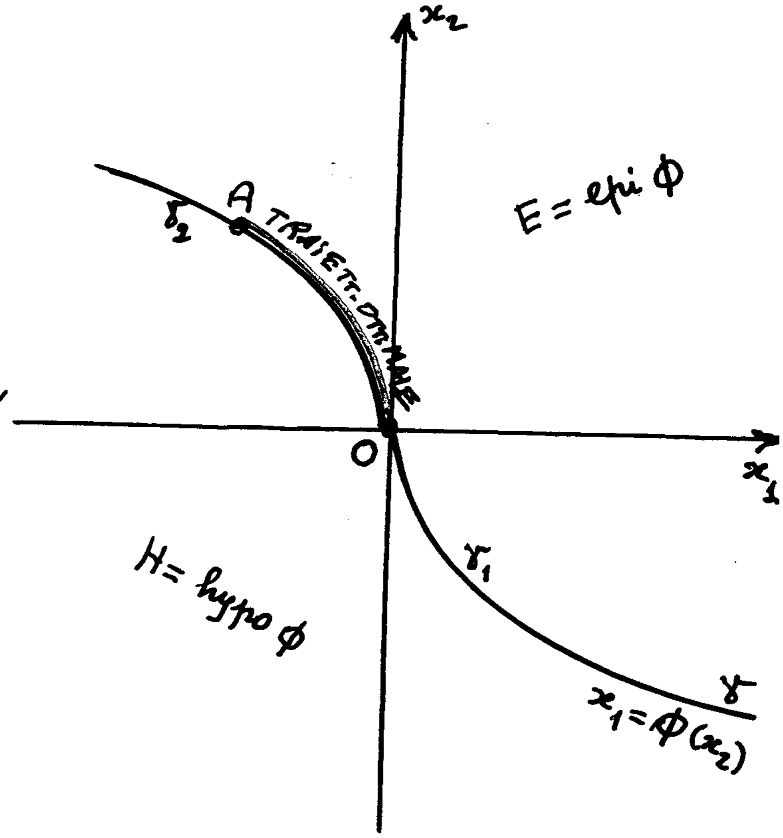
$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -1 \end{cases} \quad \begin{cases} x_1 = -\frac{1}{2}t^2 + c_5 t + c_6 \\ x_2 = -t + c_5 \end{cases} \quad \begin{cases} x_1 = -\frac{1}{2}x_2^2 \overbrace{+ \frac{1}{2}c_5^2 + c_6}^{k} \\ t = c_5 - x_2 \end{cases} \quad \boxed{x_1 = -\frac{1}{2}x_2^2 + k}$$

EQUAZIONI ORARIE

IL TEOREMA DI PONTRYAGIN AFFERMA CHE, SE UNA <u>TRAIETTORIA</u> (CONTINUA, MA NON NECESSARIAMENTE DIFFERENZIABILE), CHE DA UN PUNTO INIZIALE A PORTA NELL'ORIGINE O NEL RISPETTO DEI VINCOLI, IMPIEGA IL <u>MINIMO TEMPO</u>, ALLORA ESSA È <u>COMPOSTA DA ARCHI DELLE DUE PRECEDENTI PARABOLE</u>. CIÒ SI REALIZZA, <u>ALTERNANDO IL CONTROLLO</u> $\xi(t)$ FRA I VALORI -1 E $+1$.

POICHÈ LA TRAIETTORIA DEVE TERMINARE IN $O=(0,0)$, È OPPORTUNO INTRODURRE:

$$\Phi(x_2) := \begin{cases} \frac{1}{2}x_2^2, & \text{SE } x_2 \le 0 \\ -\frac{1}{2}x_2^2, & \text{SE } x_2 \ge 0 \end{cases}$$

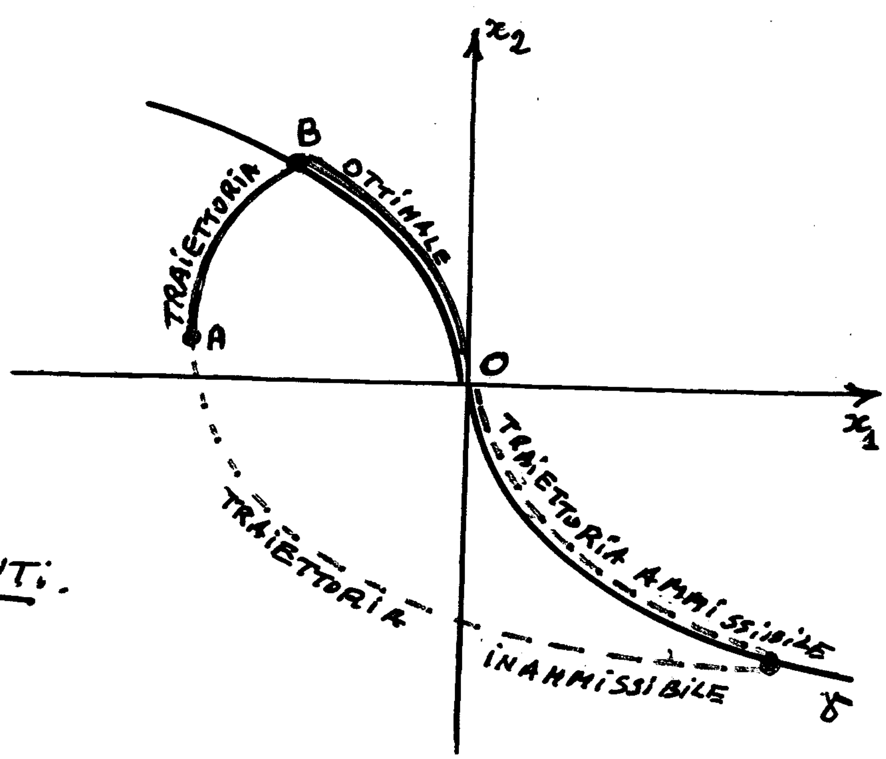


DAL TEO. PONTRYAGIN SI HA CHE, SE $A \in \gamma$, LA TRAIETTORIA OTTIMALE È L'ARCO $\overset{\frown}{AO}$ DI γ.

SE $A \notin \gamma$, ALLORA OCCORRE PERCORRERE IL PIÙ BREVE TRATTO DI UNO DEI 2 TIPI DI PARABOLE, CHE DA A PORTA IN γ, E POI PROSEGUIRE SU γ FINO AD O.

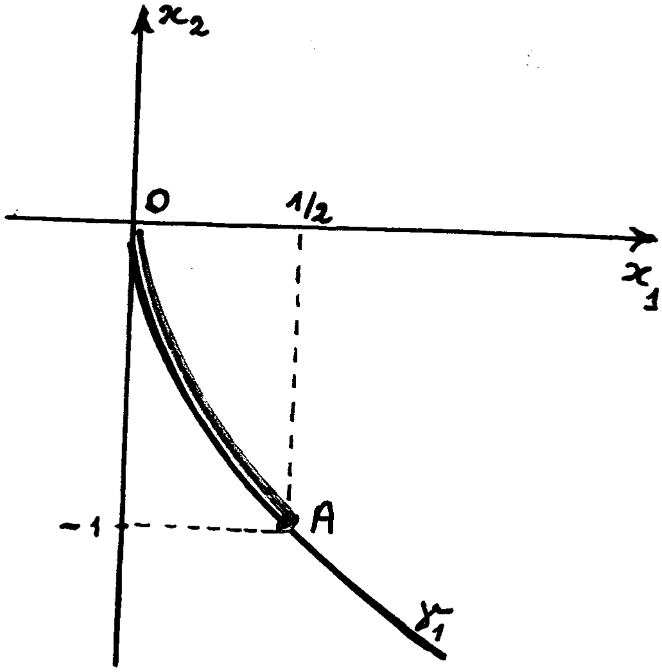
PER IL TEO. PONTRYAGIN, IL 2° VINCOLO È DIVENUTO: $\frac{dx_2(t)}{dt} = \pm 1$

QUINDI, IN UNA TRAIETTORIA OTTIMALE, UN ARCO DI PARABOLA DEL TIPO ⊕ DEVE ESSERE PERCORSO (RISPETTO AL TEMPO) NEL VERSO DELLE x_2 CRESCENTI, E DEL TIPO ⊖ NEL VERSO DELLE x_2 DECRESCENTI.



NON È RESTRITTIVO SUPPORRE $t_0=0$, COSICCHÈ IL TEMPO TOTALE DI UNA TRAIETTORIA È $t_1-t_0=t_1$.

ESEMPIO



PUNTO INIZIALE $A\equiv(\frac{1}{2},-1)\in\gamma_1 \Rightarrow k=0$.

A CORRISPONDE A $t=0$.

$$\left.\begin{matrix} k=0 \\ t=0 \end{matrix}\right\} \Rightarrow \begin{cases} -\frac{1}{2}c_3^2+c_4=0 \\ 0=-1-c_3 \end{cases} \quad \begin{cases} c_3=-1 \\ c_4=\frac{1}{2} \end{cases}$$

MUOVENDOSI DA A VERSO O, SI RISPETTA IL VERSO CRESCENTE DELLE x_2 RICHIESTO DAL CASO $(+)$.

LE EQUAZIONI ORARIE DI $(+)$ DIVENGONO:

$$\begin{cases} x_1=\frac{1}{2}t^2-t+\frac{1}{2}=\left(\frac{t}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)^2 \\ x_2=t-1 \end{cases}$$

(SONO VERIFICATE DALLE COND. INIZIALI $t=0 \quad x_1=\frac{1}{2} \quad x_2=-1$)

QUESTE, PONENDO LA CONDIZIONE FINALE $x_1=x_2=0$, PORGONO IL TEMPO TRASCORSO PER PERCORRE L'ARCO $\overset{\frown}{AO}$, CIOÈ $t_1=t=1$, CHE È ANCHE IL MINIMO TEMPO POSSIBILE.

LE EQUAZIONI PARAMETRICHE SONO

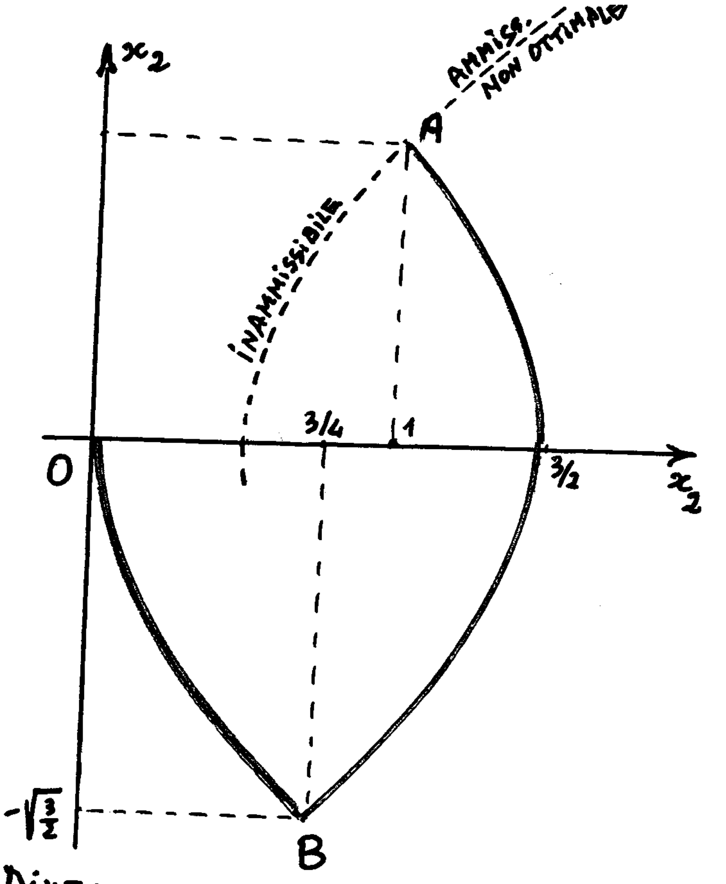
$$x_1(t)=\frac{1}{2}(t-1)^2 \qquad x_2(t)=t-1 \qquad 0\le t\le 1$$

LA SOLUZIONE DEL PROBLEMA DATO È:

$$y(t)=\frac{1}{2}(t-1), \quad 0\le t\le 1$$

IL MINIMO ESSENDO $=1$.

ESEMPIO: PUNTO INIZIALE $A \equiv (1,1)$.



IL PASSAGGIO DI PARABOLA $\ominus$ PER $A \Rightarrow k = \frac{3}{2}$;
LA PARABOLA È $x_1 = -\frac{1}{2}x_2^2 + \frac{3}{2}$.
A CORRISPONDE A $t = 0$.

$$\left.\begin{matrix} k = \frac{3}{2} \\ t = 0 \end{matrix}\right\} \Rightarrow \begin{cases} \frac{1}{2}c_5^2 + c_6 = \frac{3}{2} \\ 0 = c_5 - 1 \end{cases} \quad \begin{cases} c_5 = 1 \\ c_6 = 1 \end{cases}$$

DA A VERSO B SI RISPETTA $\frac{dx_2}{dt} = -1$.
LE EQUAZ. ORARIE DI $\ominus$ DIVENGONO:

$$\begin{cases} x_1 = -\frac{1}{2}t^2 + t + 1 \\ x_2 = -t + 1 \end{cases}$$

(SONO VERIFICATE DALLA COND. INIZIALE $t = 0 \quad x_1 = x_2 = 1$)

QUESTE, PONENDOVI LE CONDIZIONI INTERMEDIE (OTTENUTE INTERSECANDO $x_1 = -\frac{1}{2}x_2^2 + \frac{3}{2}$ ED $x_1 = \frac{1}{2}x_2^2 \Rightarrow B \equiv \left(\frac{3}{4}, -\sqrt{\frac{3}{2}}\right)$), DIVENGONO:

$$\begin{cases} \frac{3}{4} = -\frac{1}{2}t^2 + t + 1 \\ -\sqrt{\frac{3}{2}} = -t + 1 \end{cases} \quad \begin{cases} t = 1 \pm \sqrt{\frac{3}{2}} \\ t = 1 + \sqrt{\frac{3}{2}} \end{cases} \Rightarrow \underline{\text{TEMPO DA A A B} = 1 + \sqrt{\tfrac{3}{2}}} \quad \left(1 - \sqrt{\tfrac{3}{2}} < 0\right)$$

ORA SI PROSEGUE SU $\overrightarrow{BO}$, AVENDO $t = 1 + \sqrt{\frac{3}{2}}$ E B COME CONDIZ. INIZIALI.
SEGUE $h = 0$.

$$\left.\begin{matrix} h = 0 \\ t = 1 + \sqrt{\frac{3}{2}} \end{matrix}\right\} \Rightarrow \begin{cases} -\frac{1}{2}c_3^2 + c_4 = 0 \\ 1 + \sqrt{\frac{3}{2}} = -\sqrt{\frac{3}{2}} - c_3 \end{cases} \quad \begin{cases} c_3 = -\left(1 + 2\sqrt{\frac{3}{2}}\right) \\ c_4 = \frac{7}{2} + 2\sqrt{\frac{3}{2}} \end{cases}$$

(DA B VERSO O SI RISPETTA $\frac{dx_2}{dt} = 1$)

LE EQUAZ. ORARIE DI $\oplus$ DIVENGONO:

$$\begin{cases} x_1 = \frac{1}{2}t^2 - \left(1 + 2\sqrt{\frac{3}{2}}\right)t + \frac{7}{2} + 2\sqrt{\frac{3}{2}} \\ x_2 = t - \left(1 + 2\sqrt{\frac{3}{2}}\right) \end{cases}$$

(SONO VERIFICATE DALLA COND. INIZIALE $t = 1 + \sqrt{\frac{3}{2}} \quad x_1 = 3/4 \quad x_2 = -\sqrt{3/2}$)

IN ESSE PONIAMO LA CONDIZ. FINALE $x_1 = x_2 = 0$; DIVENGONO:

$$\begin{cases} 0 = \frac{1}{2}t^2 - \left(1 + 2\sqrt{\frac{3}{2}}\right)t + \frac{7}{2} + 2\sqrt{\frac{3}{2}} = \frac{1}{2}\left[t - \left(1 + 2\sqrt{\frac{3}{2}}\right)\right]^2 \\ 0 = t - \left(1 + 2\sqrt{3/2}\right) \end{cases} \Rightarrow t = 1 + 2\sqrt{\tfrac{3}{2}} \text{ DA B AD O}$$

$\underline{\text{MINIMO TEMPO DA A AD O}}$ È $t_1 = 1 + \sqrt{3/2} + 1 + 2\sqrt{3/2} = \underline{2 + 3\sqrt{\frac{3}{2}}}$

LA SOLUZIONE DEL PROBLEMA DATO È:

$$y(t) = \begin{cases} -\frac{1}{2}t^2 + t + 1, \quad \text{SE} & 0 \le t \le 1 + \sqrt{\frac{3}{2}} \\ \frac{1}{2}t^2 - \left(1 + 2\sqrt{\frac{3}{2}}\right)t + \frac{7}{2} + 2\sqrt{3/2}, \quad \text{SE} & 1 + \sqrt{\frac{3}{2}} \le t \le 2 + 3\sqrt{3/2} \end{cases}$$

IL MINIMO ESSENDO $2 + 3\sqrt{\frac{3}{2}}$