Variational Approach to the Problem of the Minimum Induced Drag of Wings

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Abstract A closed form solution of the problem of the minimum induced drag of a finite span straight wing was given by Prandtl. In this paper, a mathematical theory, based on a variational approach, is proposed in order to revise such a problem and provide one with a support for optimizing more complex wing configurations, which are becoming of interest for future aircraft. The first step of the theory consists in finding a class of functions (lift distributions) for which the induced drag functional is well defined. Then, in this class, the functional to be minimized is proved to be strictly convex; taking into account this result, it is proved that the global minimum solution exists and is unique. In Sect. Subsequently, we introduce the *Image Space Analysis* associated with a constrained extremum problem; this allows us to define the Lagrangian dual of the problem of the minimum induced drag, and show how such a dual problem can supply a new approach to the design. After having obtained the Prandtl exact solution in the context of a variational formulation, a numerical algorithm, based on the Ritz method, is presented, and its convergence is proved.

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1 Introduction

The main parts of aerodynamic drag of an aircraft are the friction and the induced ones. Friction drag depends on the wetted surface; induced drag depends on the lift distribution along the lifting systems. According to the theorem of Kutta (1867-1914) and Jukowski (1847-1921), lift equals the product of asymptotic speed, density and circulation (or vorticity). Ludwig Prandtl (1875-1953) and his collaborators at the University of Göttinghen gave a significant contribution to Aerodynamics. In the case of finite width straight wings, due to a theorem on the conservation of vorticity, the vorticity variation along the wing equals the vorticity released along the stream, which, in its turn, produces an induced velocity on the wing, in accordance with the Biot and Savart law. Prandtl gave a solution to the Variational Problem of assessing the lift distribution for which, given the total lift, the induced drag is a minimum. For this problem, the optimality condition implies a constant induced velocity along the wing span and the elliptical lifting distribution satisfy this condition. This result was fundamental in the history of aviation: all actual aircraft are designed in order to obtain an elliptical lift distribution as close as possible.

From these considerations, Prandtl's problem is considered again, with the aim of introducing an extensive mathematical analysis of the problem, taking into account recent results of the theory of constrained extremum problems, in particular, the Image Space Analysis. This will lead, among other things, to formulate a Lagrangian dual problem of that of minimum induced drag. It is shown that such a dual problem is a new approach to the primary problem.

2 Finite Span Wings

In a steady, subsonic and two dimensional stream, the aerodynamic force acting on a solid body is given by

$$D = 0, \quad P = \rho \mathbf{V}_\infty \Gamma,$$

where D is the component along the asymptotic stream direction and P is the normal to D; ρ and $\mathbf{V}_\infty$ are the density and the asymptotic velocity, respectively. This result is known as the Kutta-Jukowski Theorem and, accordingly, the drag on a profile is zero, independently of the lift. This result, even for inviscid fluids, is no longer valid when dealing with a finite span wing, where a drag induced by the lift (and, hence, named "induced drag") is present due to three-dimensional effects.

In a finite span wing (e.g., figure 1) the pressure difference between upper and lower sides produces tip horse shoe vortices which, in turn, on any wing section, induce a vertical downstream according to the well known Biot-Savart law. Thus, in any section of the wing, the angle of incidence is locally modified by an angle α_i (of induced incidence), as shown in the following figure.

With small α_i, the forces orthogonal to the stream (lift) and along the stream direction (induced drag) are:

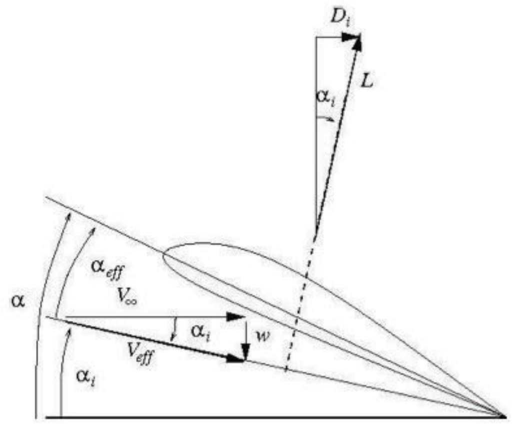

Fig. 1: α-geometric angle of incidence, α_i-angle of induced velocity, α_{eff}-actual angle of incidence.

$$\begin{cases} L = F\cos\alpha_i = \int_{-b}^{b} \rho \Gamma V_\infty dy, \\ D_i = F\sin\alpha_i = \rho\int_{-b}^{b} w(y)\Gamma dy, \end{cases} \tag{1}$$

where $2b$ is the wingspan.

In the case of a large span/cord ratio, the wing can be assumed as a lifting line undergoing a circulation distribution along the span (Prandtl). According to the Biot-Savart law, the velocity induced in y_0 by an elementary free vortex $d\Gamma = \left(\frac{d\Gamma}{dy}\right) dy$ is given by:

$$dw(y_0) = \frac{1}{4\pi}\left(\frac{d\Gamma}{dy}\frac{dy}{y_0 - y}\right), \tag{2}$$

and velocity induced by the whole vorticity becomes:

$$w(y_0) = \frac{1}{4\pi}\int_{-b}^{b}\frac{d\Gamma}{dy}\frac{dy}{y_0 - y}. \tag{3}$$

Combining the previous results, we have the induced drag:

$$D_i = \frac{\rho}{4\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{\Gamma'(x)\Gamma(y)}{y-x}dxdy.$$

Note that the double integral is defined by its principal value of Cauchy (see Sect. 6).

Fig. 2: Vortices on the wing wake: on the lifting line, the difference between vortices (or circulation) equals the free vortices detaching from trailing edge.

3 Problem of Minimum Induced Drag of a Straight Wing. An optimality condition

Denote by $\mathbb{R}$ and $\mathbb{R}_+$ the sets of reals and non-negative reals, respectively. We consider a straight wing, which is assumed to be a lifting segment of the real line; it is not restrictive to represent the segment by $T := [-1,1] \subset \mathbb{R}$ (see figure 3).

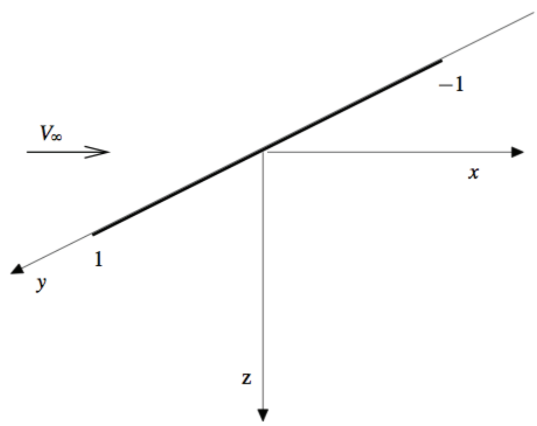

Fig. 3: Reference frame of the lifting line wing.

Let Ω be an open set of $\mathbb{R}$, such that $\Omega \supset T$. We wish to determine a function $\Gamma : \Omega \to \mathbb{R}_+$, which minimizes the induced drag, denoted by $f(\Gamma)$, subject to a constraint on the total lift, denoted by $g(\Gamma)$. Let us now give this problem a mathematical formulation. To this end, let χ be a set of functions Γ, where a solution is sought. Thus, the problem can be formulated as:

$$\min f(\Gamma) := \frac{\rho}{4\pi} \int_T \int_T \frac{\Gamma'(x)\Gamma(y)}{y-x} dxdy, \tag{4}$$

subject to

$$g(\Gamma) := \rho V_\infty \int_T \Gamma dx - c = 0, \tag{5}$$

$$\Gamma \in \chi, \tag{6}$$

where min denotes the global minimum and c is a positive constant; ":="denotes "equality by definition".

First of all, the elements of χ must make f positive and satisfies given boundary conditions $\Gamma(-1) = \Gamma(1) = 0$. Furthermore, Γ must be such that the functionals f and g exist. This is guaranteed by the assumptions of Proposition 1 of Appendix 1. Such assumptions allow us to define f on the subset of $T \times T$ where $x = y$, giving it the value of its limit, so that the functional so extended – which, without any danger of confusion, will be denoted by the same f – turns out to be continuous.

Having restricted χ in order to guarantee the existence and the continuity of f and g, we now must assure the existence of the (constrained) minimum in (4)-(6). This can be achieved in several ways. To this end, we introduce the following norm:

$$||\Gamma|| := \max_{x \in T} \{||\Gamma(x)||_2, ||\Gamma'(x)||_2\},$$

where $||\cdot||_2$ is the L^2 norm for the present problem.

A first way consists in restricting χ to be compact with respect to the above norm; a simple (but sufficient for the design of a wing) example is given by a set of bounded polynomials on T, with degree not greater than a fixed value. The assumptions of the Lebesgue Fundamental Theorem being fulfilled (Lebesgue measurability and uniform boundedness of each sequence), then the set of solutions to (5) is closed. Taking into account that a closed subset of a compact set is compact, the set of solutions to (5)-(6) is compact. This and the continuity of f give the existence of the minimum.

A second way consists in proving the strict convexity of the functional f (see Theorem 2 in sect. **??**) and show that, within the stated class χ, the first variation vanishes. Of course, this depends on the fact that the first variation vanishes on χ; otherwise, nothing can be said, unless the convexity of χ is proved; but this is not an easy task.

Condition 1 *The minimum of problem* (4)-(6) *is an increasing function of c, which tends to* $+\infty$ *as c tends to* $+\infty$.

The property expressed by the above condition, even if intuitively obvious from engineering viewpoint, is of no easy mathematical proof. Indeed, due to its importance in the analysis of the solution of (4)-(6) when c is a parameter (and not a given number) as happens in Sect. 4, we should prove it; to avoid an excessive mathematical machinery, in the sequel we will assume it.

Having discussed the existence of the minimum of problem (4)-(6), let us now consider an optimality condition. In Appendix 1 it is proved that a class, where it is suitable to look for the circulation distribution Γ as solution of the minimum problem (4)-(6), is the following:

$$\mathscr{X} = \{\Gamma \in AC[-1,1],\ \Gamma' \in \mathscr{L}^{1+\varepsilon}(-1,1), \text{ with } \varepsilon > 0,\ \Gamma(1) = 0, \\ \Gamma(-1) = 0,\ D_i(\Gamma) > 0\}.$$

Here, according to what was done by Munk [4], we can prove a necessary and sufficient optimality condition for Γ.

Theorem 1 *Let be $\Gamma \in \mathscr{X}$. Γ is solution of the isoperimetric problem* (4)-(6)*, if and only if* $w(y) = \frac{1}{4\pi}\int_{-1}^{1}\frac{\Gamma'(x)}{x-y}dx = constant,\ \forall y \in [-1,1]$.

Proof. Let Γ, $\Gamma_* \in \mathscr{X}$ and set $\delta\Gamma(x) := \Gamma - \Gamma_*$, with $||\Gamma_*|| < \varepsilon,\ \varepsilon > 0$. Introduce the functions:

$$\Gamma(z,\alpha) := \Gamma(z) + \alpha\delta\Gamma(z), \\ \Gamma'(z,\alpha) := \Gamma'(z) + \alpha\delta\Gamma'(z).$$

$\Gamma'(z,\alpha)$ is the derivative of $\Gamma(z,\alpha)$ with respect to z. Consider $J(\lambda)$ the functional:

$$J(\lambda) := \int_T\int_T \left[\frac{\rho}{4\pi}\frac{\Gamma'(x)\Gamma(y)}{y-x} - \frac{\lambda\rho V_\infty}{2}\Gamma(y)\right] dxdy\ , \tag{7}$$

and let us calculate the variation:

$$\begin{aligned} J(\lambda,\alpha) &:= \int_T\int_T \left[\frac{\rho}{4\pi}\frac{\Gamma'(x,\alpha)\Gamma(y,\alpha)}{y-x} - \frac{\lambda\rho V_\infty}{2}\Gamma(y,\alpha)\right] dxdy \\ &= \int_T\int_T \frac{\rho}{4\pi}\frac{(\Gamma'(x)+\alpha\delta\Gamma'(x))(\Gamma(y)+\alpha\delta\Gamma(y))}{y-x}dxdy + \\ &\quad - \int_T\int_T \frac{\lambda\rho V_\infty}{2}(\Gamma(y)+\alpha\delta\Gamma(y))dxdy. \end{aligned}$$

When the derivative of J with respect to α is evaluated in zero, we find:

$$J'(\lambda,0)=\int_T\int_T\left[\frac{\rho}{4\pi}\frac{\delta\Gamma'(x)\Gamma(y)+\delta\Gamma(y)\Gamma'(x)}{y-x}-\frac{\lambda\rho V_\infty}{2}\delta\Gamma(y)\right]dxdy$$

$$=\int_T\int_T\left[\frac{\rho}{4\pi}\frac{\delta\Gamma(y)\Gamma'(x)}{y-x}-\frac{\lambda\rho V_\infty}{2}\delta\Gamma(y)\right]dxdy+ \tag{8}$$

$$+\int_T\int_T\frac{\rho}{4\pi}\frac{\delta\Gamma'(x)\Gamma(y)}{y-x}dxdy.$$

Integrating by parts the second term of equation (8) leads to:

$$\begin{aligned}\frac{\rho}{4\pi}\int_T\int_T\frac{\delta\Gamma'(x)\Gamma(y)}{y-x}dxdy=&\left[\frac{\rho}{4\pi}\delta\Gamma(x)\int_T\frac{\Gamma(y)}{y-x}dy\right]_{-1}^{1}\\&-\frac{\rho}{4\pi}\int_T\delta\Gamma(x)\frac{d}{dx}\int_T\frac{\Gamma(y)}{y-x}dxdy.\end{aligned} \tag{9}$$

The first term in the right-hand side of equation (9) is null because
$\delta\Gamma(-1)=\delta\Gamma(1)=0$. Having put $t=y-x$, we have:

$$\frac{d}{dx}\int_T\frac{\Gamma(y)}{y-x}dy=\frac{d}{dx}\int_T\frac{\Gamma(t+x)}{t}dt=\int_{-1-x}^{1-x}\frac{\Gamma'(t+x)}{t}dt.$$

Coming back to the previous variables, we find:

$$\frac{d}{dx}\int_T\frac{\Gamma(y)}{y-x}dx=\int_T\frac{\Gamma'(y)}{y-x}dy. \tag{10}$$

By exchanging y with x in the right-hand side of equation (9) and remembering (10) the right-hand side of (8) becomes:

$$J'(\lambda,0)=\int_T\int_T\delta\Gamma(y)dy\left(\int_T\left(\frac{\rho}{2\pi}\frac{\Gamma'(x)}{y-x}-\frac{\lambda\rho V_\infty}{2}\right)dx\right). \tag{11}$$

Since $J'(\lambda,0)=0$, a sufficient condition is:

$$w(y)=\frac{1}{4\pi}\int_T\frac{\Gamma'(x)}{x-y}dx=costant,\ \ \forall y\in[-1,1]. \tag{12}$$

This condition is necessary as well, after observing that the functional is convex; in fact, the second derivative of J with respect to α is:

$$J''(\lambda,\alpha)=\frac{\rho}{4\pi}\int_T\int_T\frac{\delta\Gamma'(x)\delta\Gamma(y)}{y-x}dxdy, \tag{13}$$

where the quantity at the right-hand side is the elementary induced drag D_i , due to the lift and, therefore, is positive. □

4 Duality. A New Approach to the Design of Wings

Now, we want to introduce the dual problem of (4)-(6). To this end, let us say first of all something about the birth of duality.

A general feature of duality (it would be better to say dualism) consists in two entities, which express a sort of symmetry or complementary. In the field of Optimization, such entities are a pair of constrained extremum problems. An early trace of this – perhaps, the first – is due to Vecten, and, independently, to Fasbender (see [1], [2]) with reference to Fermat–Torricelli problem on a triangle (which consists in finding a point of a triangle – now called Torricelli point –, which minimizes the sum of its distances from the vertices; the given problem is called *primal*): among all the equilateral triangles, which are circumscribed to a given triangle, to find one having maximum height; they showed that such a maximum height equals the minimum sum of the distances of Torricelli point from the vertices of the given triangle (see [1], page 235). Fermat–Torricelli and Vecten–Fasbender problems are a pair of constrained extremum problems, which enjoy the following properties:

(i) they are defined by the same data;
(ii) they search for opposite extrema;
(iii) the values of their objective functions, corresponding to feasible solutions, form two sets of real numbers, which are separated;
(iv) the two extrema are equal; the common value of the two extrema being, therefore, the separating element of two contiguous classes of real numbers.

The above problems enjoy further properties, which Vecten and Fasbender did not observe (and, perhaps, could not have noted at that time):

(v) *relaxation*: the dual is equivalent to search, in the primal, for the best lower bound of the objective function obtained by *relaxing* the feasible region (of course, if the primal searches for the maximum – like in the problem of this paper –, then relaxation must be replaced by *contraction*);
(vi) *reflexivity*: the dual of the dual problem is (equivalent to) the primal.

The result by Vecten and Fasbender has marked the birth of duality theory for constrained extremum problems. Subsequently, a few results appeared till when John von Neumann claimed the above (i)–(iv) for a linear programming problem. After von Neumann result, the theory of duality grew quickly; it achieved the present general form, when it was recognized to be a step of the Image Space Analysis carried on through Hahn–Banach separation theory. Appendix 2 contains a short outline of Image Space Analysis and how it can lead to discover the theory of duality. Here, by a logic-intuitive way, we merely consider the essentials steps to achieve the dual problem of (4)-(6) which, for symmetry of language, is called *primal*.

Denote by $R := \{\Gamma \in \chi : \rho V \int_T \Gamma(x) - c = 0\}$ the feasible region of (4)-(6). Let us start with the obvious remark that $\overline{\Gamma} \in R$ is a (global) minimum point of (4)-(6), iff the system in the unknown Γ; the notation is the same as in Sect. 5)

$$u := f(\overline{\Gamma}) - f(\Gamma) > 0, \quad v := g(\Gamma) = 0, \quad \Gamma \in \chi \tag{14}$$

is impossible. u and v run in the images of χ through the functionals $f(\overline{\Gamma}) - f$ and g, respectively. Therefore, the space where Γ runs is paired with $\mathbb{R}^2$ where (u,v) runs; this $\mathbb{R}^2$ is called the *Image Space* associated with (4)-(6); the set:

$$\mathscr{K}_{\overline{\Gamma}} := \left\{(u,v) \in \mathbb{R}^2 : u = f(\overline{\Gamma}) - f(\Gamma), v = g(\Gamma), \Gamma \in \chi\right\}$$

is called the iimage set of (4)-(6). By introducing the set

$$\mathscr{H} := \left\{(u,v) \in \mathbb{R}^2 : u > 0, v = 0\right\},$$

which mirrors the conditions in (14), we can say that $\overline{\Gamma} \in R$ is a (global) minimum point of (4)-(6), iff

$$\mathscr{H} \cap \mathscr{K} = \emptyset. \tag{15}$$

It is trivial to note that (14) is impossible, iff (15) holds. While (14) has an algebraic appeal, (15) appears a geometrical approach. In fact, the disjunction (15) can be proved, by showing that there exists a line (of $\mathbb{R}^2$), say H^0, such that $\mathscr{H}$ and $\mathscr{K}$ lie in the halfplanes, say H^+ and H^-, the former open and the latter closed, defined by H^0, respectively. Thus, taking into account that $\mathscr{H}$ is a halfline (of $\mathbb{R}^2$) desprived of the vertex and $\mathscr{K}$ can be replaced equivalently by a convex set (see Appendices 1, 2), and defining H^0, H^- and H^+, respectively, by:

$$\theta u + \lambda v = 0, \quad \theta u + \lambda v \le 0, \quad \theta u + \lambda v > 0, \quad \theta, \lambda \in \mathbb{R}, \tag{16}$$

it is easy to note that (15) is equivalent to the existence of $\overline{\lambda} \in \mathbb{R}$, such that:

$$\overline{\theta} u + \overline{\lambda} v \le 0, \quad \forall (u,v) \in \mathscr{K}. \tag{17}$$

Due to the homogeneity of the inequalities (16) and (17), we might set $\theta = 1$; this is not done, because of the meaning that θ and λ will have in the subsequent application. In other words, $\mathscr{H}$ being included by definition in H^+, (15) holds if and (because of the convexity of both $\mathscr{H}$ and $\mathscr{K}$) only if (17) holds. Now, by recalling the definition of u and v in (14), (17) turns out to be equivalent to the existence of $\overline{\theta} > 0$ and $\overline{\lambda}$, such that:

$$\mathscr{L}(\overline{\Gamma}; \overline{\theta}, \overline{\lambda}) \le \mathscr{L}(\Gamma; \overline{\theta}, \overline{\lambda}), \quad \forall \Gamma \in \chi, \tag{18}$$

where

$$\mathscr{L}(\Gamma; \theta, \lambda) := \int_T \int_T \left[\theta \frac{\rho}{4\pi} \frac{\Gamma'(x)\Gamma(y)}{y - x} - \lambda \rho V_\infty \frac{\Gamma(x)}{2}\right] dxdy. \tag{19}$$

is the *Lagrangian function*.

In fact, $\overline{\Gamma} \in R$ implies $g(\overline{\Gamma}) = 0$, so that $\mathscr{L}(\Gamma; \overline{\theta}, \overline{\lambda}) = \theta f(\overline{\Gamma})$ and the inequality (18) is equivalent to (17).

We have thus shown that the fulfillment of inequality (18) is equivalent to prove the optimality of $\overline{\Gamma}$. However, to verify (18) is not an easy task. Therefore, in order to prove (18), we are led to evaluate, for each (θ, λ), the minimum (in general, infimum) of $\mathscr{L}(\Gamma; \theta, \lambda)$ with respect to $\Gamma \in \chi$ (and not $\Gamma \in R$), and then the maximum

(supremum, in general) of such minima (which, obviously, depends on θ, λ) with respect to θ, λ.
In other words, we are led to introduce the following problem:

$$\max_{\theta > 0, \lambda \in \mathbb{R}} \min_{\Gamma \in \chi} \mathscr{L}(\Gamma; \theta, \lambda), \tag{20}$$

which is called *dual problem* of (4)-(6). Due to special properties of (4)-(6), it is possible to prove that (20) enjoys the properties (i)-(iv); see Appendix 2.

The Image Space Analysis allows one to achieve several other important properties and informations; see Appendix 2.
In particular, we can draw that a solution $(\overline{\Gamma}; \overline{\theta}, \overline{\lambda})$ of the dual problem (19) enjoys this property: $\overline{\lambda}/\overline{\theta}$ is nothing more than the classic Lagrangian multiplier and *allows one to change in the minimum induced drag consequent to a change in the value at which the total lift is constrained*.

Now, we are in the position to discuss a different approach to the design of the wing. As discussed in details in Appendix 2, to consider the induced drag as an objective (to be minimized) and the total lift as a constraint is absolutely subjective. An alternative approach, which does not oblige us to consider one of the two entities as constraint, is the following. *We consider both entities as objectives*, in the sense that *we aim to minimize the induced drag and to maximize the total lift* or, equivalently, to minimize the opposite of the total lift. To try *to fulfil both objectives, we consider a combination of them*:

$$\theta f(\Gamma) - \lambda g(\Gamma), \quad \Gamma \in \chi, \tag{21}$$

where $\theta, \lambda > 0$. *If we minimize* (21) *(depending on the weights* θ *and* λ*, the minimum may not exist, and the infimum may by* $-\infty$*), we certainly pursue both objectives, even if through a mixture of them*. However, by itself, such a minimization does not give us any guarantee, until we identify (21) with (19) and we exploit the previous analysis. This way, we discover that the minimum of (21), which is obviously a function of the "weights"of the combination, is $\leqslant$ of that of (4)-(6). In other words, *by minimizing such a combination, we find a lower bound of the minimum of* (4)-(6). This result is rather intuitive: in setting up a combination of the objectives and, in addition, choosing arbitrary "weights", *we have been "arbitrarily optimistic"as concerns the design of the wing*. At this point, it comes natural *to search, among all such "optimistic designs", for one which is the least optimistic*; in other words, we look for the maximum, with respect to the "weights", of the several above minima (found with respect to $\Gamma \in \chi$), of (21). But this is the dual problem of (4)-(6), so that we obtain the same result as from (4)-(6) (see Theorem 4 of Appendix 2 for details).

The dual problem of (4)-(6), namely (20) (see also the 1st side of (19) of Appendix 2) depends on the constant c, even if it does not appear explicitly in (20). Now, replace c with the parameter ξ and denote the dual problem by $P^*(\xi)$; in other words, $P^*(\xi)$ is the dual of (4)-(6), where c is replaced by ξ. Let us now continue the interpretation of $P^*(\xi)$ and its exploitation for the design. By solving $P^*(\xi)$, we find the functions:

$$\Gamma(\xi), \quad \theta(\xi), \quad \lambda(\xi), \quad f^*(\xi), \tag{22}$$

the last of which gives the maximum in (20) (or in the 1st side of (61) of Appendix 2). As said before for the case $\xi = c$, the ratio $\lambda(\xi)/\theta(\xi)$ gives a fundamental information for the design (see Appendix 2 for details). Thus, it is reasonable to assume that the designer can define a function, say $\varphi : [c_1, c_2] \to \mathbb{R}_+$ with c_1, c_2 given positive constants within which ξ must lie, which expresses a measure of the merit for the project consequent to the value of the ratio $\lambda(\xi)/\theta(\xi)$. It is reasonable to suppose also that φ be unimodal (so that it possesses maximum and unique maximum point). Then, the designer can now consider the problem:

$$\max_{\xi \in [c_1, c_2]} \varphi\left(\frac{\lambda(\xi)}{\theta(\xi)}\right). \tag{23}$$

By solving it, he finds the unique maximum point, say $\overline{\xi}$. Consequently, with regard to the combination (21) of the two objectives, $\theta(\overline{\xi})$ and $\lambda(\overline{\xi})$ are "the best weights" with respect to the minimax criterion (expressed by the dual problem) and that is expressed by the merit function φ. This way, the designer avoids to perform an empirical choice of the weights. For such an approach, the Image Space Analysis (see Appendix 2) has been instrumental: to see this, it is enough to note that the pair $(\theta(\xi), \lambda(\xi))$ is the gradient of a supporting line of the image set (or its conic extension) of (4)-(6) (see Appendix 2, Definition 2); the functional form of this line is the core of the dual problem.

The above approach can be generalised in several ways. First of all, we can be faced with $l \geq 2$ objectives. In this case, by exploiting the reciprocity principle mentioned in (59) of Appendix 2, we can limit ourselves to consider $l-1$ ratios of type λ/θ. In order to formulate (4)-(6), we have chosen the constant c; in as much as such a constant is replaced by a parameter, the determination of c becomes no longer essential. When the determination of the functions (22) may be computationally complex, the global analysis can be replaced by a local one.

To sum up the previous development, we can observe that the problem takes the remarkable role to free us from the irrational task to choose arbitrary (or with an empirical criterion) the weight of a combination of objectives. Of course, this fact can be generalized in various directions, in particular, when there are more than two entities/objectives; in such a case, the choice, among several entities, of one to be considered as objective may be much more difficult than in the present case of only two entities.

As concerns the computational aspects, let us observe that, due to Proposition 7, the dual problem of (4)-(6) can be equivalently reduced to just one operation:

$$\max_{\Gamma \in \chi; \theta, \lambda > 0} \mathscr{L}(\Gamma; \theta, \lambda), \quad \text{subject to} \quad \mathscr{L}'_{\Gamma}(\Gamma; \theta, \lambda) = 0, \tag{24}$$

where $\mathscr{L}'_{\Gamma}$ denotes the first variation of $\mathscr{L}$ with respect to Γ.

The second side of (61) offers a further interpretation in terms of designing a wing. For each design $\Gamma \in \chi$, we consider again the combination (21), but, this time, we keep fixed Γ and we maximize it with respect to $\theta, \lambda > 0$ (depending on Γ, the maximum may not exist, and the supremum may be $+\infty$). This way, each design

Γ is associated with a maximum of (21); this means to be "arbitrarily pessimistic". Then, by minimizing such a result with respect to Γ, we look for the least pessimistic situation and free ourselves from the arbitrariness of the choice of the weights.

Let us now give a concise interpretation of the above development. The optimality of the circulation distribution $\overline{\Gamma}$ has been reduced to show separation, by means of a line, between two sets, the image set $\mathscr{K}_{\overline{\Gamma}}$ and $\mathscr{H}$. The separation line, namely H^0 (which in Appendix 2 is denoted, with a better notation, by $H^0(\overline{\theta},\overline{\lambda},0)$), turned out to be a support line of the image set. The gradient of such a line, namely $(\overline{\theta},\overline{\lambda})$, has shown to provide us with an extremely important information about the given problem: indeed, $\overline{\lambda}/\overline{\theta}$ is the so–called Lagrange multiplier and is the (instantaneous) velocity with which the minimum induced drag changes with respect to the total lift (up to a constant). Hence, the dual problem of (4)-(6) can be viewed as the search for "the best"among the support lines of the image set, or H^0. Thus, a spontaneous remark may arise: such a support line is not contained in the data which define (4)-(6); being an adjunctive entity, which comes from the exterior of (4)-(6), the line should have not an importance and be a mere catalyst; indeed, the support line is a tangent (Bouligand tangent, if at the supporting point the image set is not smooth), and this explains in a straightforward way its importance.

Another aspect of the above development has consisted in providing us with a way of proving the existence of the minimum of (4)-(6), which is much easier than the classic one: in fact, a remarkable fact is that such a way requires to us to prove the existence of the extremum of a problem in a finite dimensional space, namely the IS which in the present case is the Euclidean plane, notwithstanding the fact that the given problem is infinite dimensional (Γ runs in a Banach space), while the classic ones require to prove the existence of the extremum in an infinite dimensional space (just that Banach space).

To sum up some of the wonderful aspect of the duality theory, we can say that the dual problem allows us:

(j) to achieve important theoretical, analytical results;
(jj) to improve solving methods for the given problem;
(jjj) to obtain, with the dual variables, a knowledge on the given problem which, often, is more important than the solution itself of the given problem; for instance, if the given problem represents an engineering design, often its solution is not striking for the designer and merely refines what he already knows; on the contrary, almost always, the solution of the dual problem brings a precious and *unexpected* information or even leads to a new approach to the design, as shown in this subsection.

5 Direct Methods

In this section, we determine the solution, Γ, of the isoperimetric problem, and propose a computation direct method to obtain a set of approximations to Γ, converging to Γ. In this method, the Γ circulation is obtained by means of the two following procedures:

- a classic Fourier expansion of Γ;
- the Ritz method, with two minimizing sequences of the type

$$\Gamma_n(x) = \sum_{i=0}^{n} b_i W_i(x), \quad n \in \mathbb{N},$$

where $W_i = (1-x^2)x^i$ in the former type and $W_i = (1-x^2)^i$ in the latter one.

5.1 Elliptic Distribution

We put $y = \cos\theta$ and, hence $dy = -\sin\theta$, and consider the Fourier expansion of Γ, with the conditions $\Gamma(-1) = \Gamma(1) = 0$, or:

$$\Gamma = \sum_{n=1}^{\infty} a_n \sin(n\theta). \tag{25}$$

The expression of lift L becomes:

$$\begin{aligned} L &= \rho V_\infty \int_{-1}^{1} \Gamma(y)dy \\ &= \rho V_\infty \int_{0}^{\pi} \Gamma(\theta)\sin\theta d\theta \\ &= \rho V_\infty \sum_{n=1}^{\infty} a_n \int_{0}^{\pi} \sin\theta \sin(n\theta)d\theta \\ &= \rho V_\infty \left(a_1 \int_{0}^{\pi} \sin^2\theta d\theta + \sum_{n=2}^{\infty} a_n \int_{0}^{\pi} \sin\theta \sin(n\theta)d\theta \right). \end{aligned}$$

Because $\int_0^\pi \sin(m\theta)\sin(n\theta)d\theta = 0$ if $n \neq m$, we have:

$$L = \rho V_\infty a_1 \int_{0}^{\pi} \sin^2\theta d\theta = \frac{\pi}{2} a_1 \rho V_\infty. \tag{26}$$

In $y_0 \in [-1,1]$ the induced velocity is:

$$w(y_0) = \frac{1}{4\pi} \int_{-1}^{1} \frac{d\Gamma(y)}{dy} \frac{1}{y-y_o} dy, \tag{27}$$

or, equivalently:

$$
\begin{aligned}
w(\theta_0) &= -\frac{1}{4\pi}\int_0^{\pi}\frac{d\Gamma(\theta)}{d\theta}\frac{1}{\cos\theta-\cos\theta_o}d\theta \\
&= -\frac{1}{4\pi}\sum_{n=1}^{\infty} na_n \int_0^{\pi}\frac{\cos(n\theta)}{\cos\theta-\cos\theta_o}d\theta.
\end{aligned}
\tag{28}
$$

Due to the Glauert formula we obtain:

$$
\begin{aligned}
w(\theta_0) &= -\frac{1}{4\pi}\sum_{n=1}^{\infty} na_n\left(\pi\frac{\sin(n\theta_0)}{\sin\theta_0}\right) \\
&= -\frac{1}{4}\sum_{n=1}^{\infty} na_n\frac{\sin(n\theta_0)}{\sin\theta_0},
\end{aligned}
\tag{29}
$$

and, finally,

$$
\begin{aligned}
D_i &= -\frac{\rho}{4}\int_0^{\pi}\left(\sum_{n=1}^{\infty} na_n\frac{\sin(n\theta)}{\sin\theta}\right)\left(\sum_{n=1}^{\infty} a_n\sin(n\theta)\right) - \sin\theta d\theta \\
&= \frac{\rho}{4}na_n^2\int_0^{\pi}\sin^2(n\theta)d\theta = \frac{\rho\pi}{8}(a_1^2+2a_2^2+\ldots+na_n^2+\ldots)
\end{aligned}
\tag{30}
$$

Because all the terms are positive and, in order to have a non-negative lift, we need $a_1 \neq 0$, the induced drag is minimum when $a_n^2 = 0,\ \forall n > 1$. Putting $a_1 = \Gamma_0$ the solution of the isoperimetric problem (4)-(6) is:

$$
\Gamma(\theta) = \Gamma_0\sin\theta, \tag{31}
$$

or, in terms of y:

$$
\Gamma(y) = \Gamma_0\sqrt{1-y^2}, \tag{32}
$$

and the induced drag D_i becomes:

$$
D_i = \frac{\rho\pi}{8}\Gamma_0. \tag{33}
$$

Remarks 1 *When: $\Gamma(\theta) = \Gamma_0\sin\theta$, then:*

- *the induced velocity w is constant, because:*

$$
w(\theta) = -\frac{1}{4\pi}\int_0^{\pi}\frac{d\Gamma(\alpha)}{d\alpha}\frac{1}{\cos\alpha-\cos\theta}d\alpha = -\frac{\Gamma_0}{4} = costant; \tag{34}
$$

- *taking equations (26) and (33) into account, when Γ is elliptical, it is trivial to obtain the well known result in Aerodynamics:*

$$
D_i = \frac{L^2}{2\pi\rho V_\infty}. \tag{35}
$$

5.2 Ritz Method

In this subsection, we obtain an approximate solution of the isoperimetric problem (4)-(6) by means of the Ritz method. The unknown circulation shape functions are of the following type

$$\Gamma_n := \sum_{i=0}^{n} b_i W_i(x), \quad n \in \mathbb{N},$$

where, for example,

$$W_i(x) = b_i(1-x^2)^{i+1} \text{ and } W_i(x) = b_i(1-x^2)x^i,$$

in order to satisfy the kinematic boundary conditions $\Gamma_n(-1) = \Gamma_n(1) = 0$.

The two classes of polynomials are indicated as TIPO1 and TIPO2 respectively; both of them respect the boundary conditions $W_i(1) = W_i(-1) = 0$.

We remark that, even though polynomials TIPO1 are symmetric and TIPO2 are not, we do not need to assume any condition of symmetry from physics, because symmetry is intrinsic in the mathematical solution of the isoperimetric problem; in fact, for any Γ_n of TIPO2, the optimum solutions give $b_i = 0$, for all i odd.
Now we describe a generic iteration for Γ_n :

5.2.1 Algorithm

- We write the induced drag D_i as a function of Γ_n, by solving the double integral according to the principal value of Cauchy; moreover, because $\Gamma_n = \sum_{i=0}^{n} b_i W_i(x)$, $n \in \mathbb{N}$, at any step we know $D_i(f_{n-2})$, relevant to the previous one. We obtain:

$$D_i(\Gamma_n) = D_i(b_0,\dots,b_n) = D_i(f_{n-2}) + \frac{\rho}{4\pi} \sum_{i=n-1}^{n} \sum_{s=0}^{n} b_i b_s \int_{-1}^{1}\int_{-1}^{1} \frac{W_i(y)W_s'(x)}{y-x} dxdy +$$
$$+ \frac{\rho}{4\pi} \sum_{i=0}^{n-2} \sum_{s=n-1}^{n} b_i b_s \int_{-1}^{1}\int_{-1}^{1} \frac{W_i(y)W_s'(x)}{y-x} dxdy.$$

  Because the functional D_i is quadratic with respect to Γ_n and Γ_n', the result of the integration is a second order homogeneous polynomial in b_i, $i = 0,\dots,n$.
- The lift is written as a function of Γ_n as well, and we have:

$$L(\Gamma_n) = L(b_0,\dots,b_n) = \rho V_\infty \sum_{i=0}^{n} b_i \int_{-1}^{1} W_i(y)dy.$$

  The function L is linear in the unknowns b_i. The isoperimetric problem (4)-(6) becomes:

$$(P) \quad \min D_i(b_0,\dots,b_n), \quad \text{s.t.} \quad L(b_0,\dots,b_n) = c, \quad (b_0,\dots,b_n) \in \mathbb{R}^{n+1}. \quad (36)$$

- The functional induced drag is a second order homogeneous polynomial in b_i, and the condition that the derivatives of the Lagrangian with respect to b_i must be zero is equivalent to solve the following linear system:

$$Ax = b$$

  where:

  - the matrix A of coefficients is the Hessian of the Lagrangian

$$J(b_0, \ldots,\ b_n, \lambda) = D_i(b_0, \ldots, b_n) - \lambda L(b_0, \ldots, b_n);$$

  - $b = [0, \ldots, 0, -c]$;
  - Because of the convexity of the functional induced drag it results that $x = [\overline{b_0}, \ldots, \overline{b_n}, \overline{\lambda}]$ is a point of minimum of the Lagrangian J.

- Once the n+1-th $(\overline{b_0}, \ldots, \overline{b_n})$ we calculate $D_i(\overline{b_0}, \ldots, \overline{b_n})$ and, hence, D_i as a function of c:

$$D_i(\overline{b_0}, \ldots, \overline{b_n}) = \alpha_n c^2. \tag{37}$$

- Calculation of the Oswald coefficient ”e” , that is :

$$e := \frac{\overline{D_i}}{D_i(\overline{b_0}, \ldots, \overline{b_n})}, \tag{38}$$

  where $\overline{D_i}$ is the induced drag relevant to the elliptic circulation defined in (35):

The algorithm described before has been implemented by using the commercial code MapleV, with a symbolic computation.
As an example, we apply the iterative procedure with the following conditions:

- $\rho = 1$;
- $V_\infty = 1$;
- $c = 100$;
- elliptical circulation $\overline{\Gamma} = 63.6942675\sqrt{1-x^2}$;
- $\overline{D_i}$ corresponding to $c = 100$ is worth: 1592.356688;
- let us indicate $\overline{\Gamma}_{max}$ the maximum value of $\overline{\Gamma}$ inside $[-1, 1]$.

5.2.2 Numerical results

In this section, some numerical results are reported in order to show that the method is convergent when “n”becomes larger and larger.

Example 1

$$n = 8$$

Γ_8 is shown, for TIPO1 polynomials, in Figure 4:

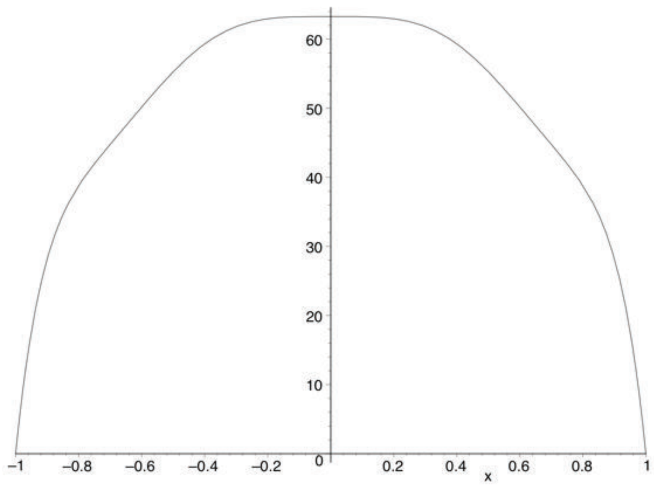

Fig. 4: Circulation distribution Γ_8

$$D_i(\Gamma_8) = 1612.261147$$
$$e = 0.9876543207$$

The error when $\overline{\Gamma}$ is approximated with $\Gamma_8(x)$ is shown in figure 5.

Example 2

$$n = 44$$

Γ_{44} is shown in Figure 6:

$$D_i(\Gamma_{44}) = 1593.143425$$
$$e = 0.9995061731$$

Figure 7 shows, for any $x \in [-1,1]$, the error when $\overline{\Gamma}$ is approximated with $\Gamma_{44}(x)$.

$$D_i(\Gamma_{44}) = 1594.252351, \quad e = 0.9988109392$$

In the case of TIPO2 polynomials the results are very similar and they are not reported here for brevity sake. The results show that the iterative procedure converges to the exact solution and, also, that the solutions of TIPO1 and TIPO2 polynomials (with the same degree) give the same induced drag. As an example, Table 1 shows the TIPO1 main data relevant to some iterations.

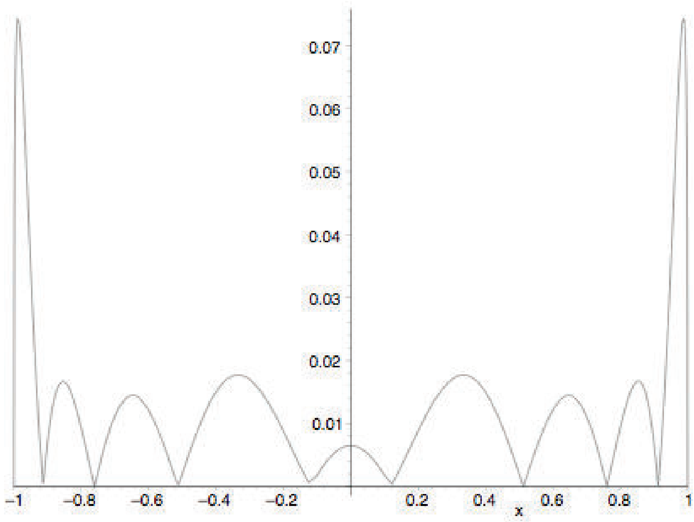

Fig. 5: Function $y(x) = \frac{|\Gamma_8(x) - \overline{\Gamma}(x)|}{\overline{\Gamma}_{max}}$.

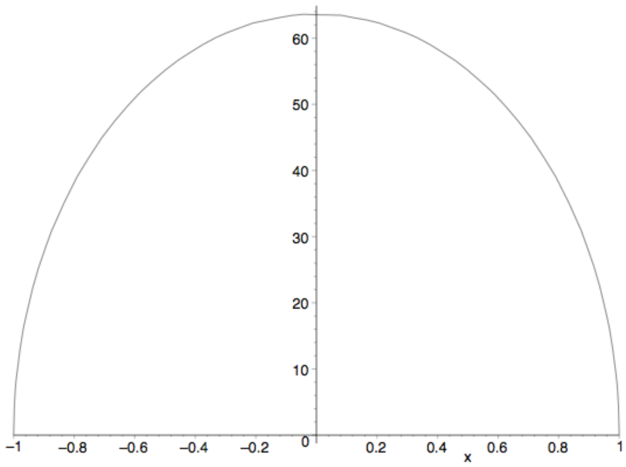

Fig. 6: Circulation distribution Γ_{44}

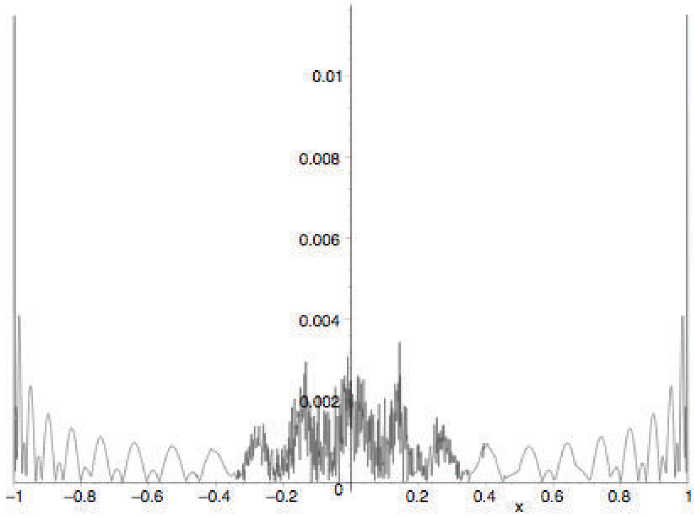

Fig. 7: Function $y(x) = \frac{|\Gamma_{44}(x) - \overline{\Gamma}(x)|}{\overline{\Gamma}_{max}}$.

Table 1: Numerical iterations relevant to TIPO1 polynomials.

Degree	D_i	$\frac{D_i - \overline{D_i}}{\overline{D_i}}$	e
4	1658.704883	0.04166666646	0.9600000002
12	1601.835002	0.005952381192	0.9940828400
20	1595.975680	0.002272726976	0.9977324266
28	1594.252351	0.001190476364	0.9988109392
36	1593.520691	0.0007309938840	0.9992695401
44	1593.020170	0.0004166666960	0.9995835068
52	1592.923767	0.0003561256120	0.9996440012
64	1592.733666	0.0002367421840	0.9997633138

References

1. V. Boltyanski, H. Martini and V. Soltarr: *Geometric Methods and Optimization Problems.* Kluwer Academic Publishers, Dordrecht, 1999.
2. F. Giannessi: *Constrained optimization and Image Space Analysis. Vol. 1: Separation of Sets and Optimality Conditions.* Springer, New York, 2005.
3. F. Giannessi: *On the theory of Lagrangian duality.* Optimization Letters, Vol. 1, pp 9–20, Springer, New York, 2005.
4. Munk M.: The minimum induced drag in airfoils, NACA 121(1924).
5. Munk M.: Isoperimetrische Aufgaben aus der Theorie des Fluges, Inaugural Dissertation 1919, Gottinga (1919).
6. Prandtl L.: Induced Drag of Multiplanes, NACA TN 182 (1924).

6 Appendix 1. Existence and convexity of the Induced Drag functional

Before proving the existence of the functional Induced Drag, we recall some useful definitions.

Definition 1 *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous in $[a,b]$, and we write $f \in AC[a,b]$ iff, for any $\varepsilon > 0$ it exists $\delta > 0$ such that for any finite collections of disjoint intervals $]\alpha_i, \beta_i[$, $i = 1,\ldots,k$, included in $[a,b]$ e with $\sum_{i=1}^{k}(\beta_i - \alpha_i) < \delta$, it results $\sum_{i=1}^{k} |f(\beta_i) - f(\alpha_i)| < \varepsilon$.*

Definition 2 *Let $(Y,\mathscr{F},\mu)$ be a measure space and $1 \le p < \infty$. We put $\mathscr{L}^p(Y) = \{f : Y \to \overline{\mathbb{R}} :\ f$ is measurable and $\int_Y |f|^p d\mu < \infty\}$.*

If q is conjugate exponent of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and, by stipulation, the conjugate exponent of 1 is ∞ and viceversa), we have $||f||_{\mathscr{L}^p(Y)} = \left[\int_Y |f|^p d\mu\right]^{\frac{1}{p}}$.

Hölder Inequality. If $f \in \mathscr{L}^p(Y)$ e $g \in \mathscr{L}^q(Y)$, then $fg \in \mathscr{L}^1(Y)$ and $||fg||_1 \le ||f||_p ||g||_q$.

Proposition 1 *Let $f \in AC]-1,1[$ be such that:*

$$f(1) = f(-1) = 0, \quad f' \in \mathscr{L}^{1+\varepsilon}]-1,1[, \textit{with } \varepsilon > 0.$$

Then

$$\int_{-1}^{1}\int_{-1}^{1} \frac{f'(x)f(y)}{y-x} dxdy$$

is convergent as a Cauchy improper integral.

Proof. Let us set:

$$S_1(h) := \{(x,y) \in \mathbb{R}^2 :\ x+h < y < 1,\ -1 < x < 1-h\}\ ,$$
$$S_2(h) := \{(x,y) \in \mathbb{R}^2 :\ -1 < y < x-h,\ -1+h < x < 1\}\ ,$$
$$G_{S_i(h)}(f) := \int\int_{S_i(h)} \frac{f'(x)f(y)}{y-x} dxdy, \quad i = 1,2.$$

Let us integrate by parts both $G_{S_1(h)}$ and $G_{S_2(h)}$:

$$G_{S_1(h)}(f) = \int_{-1}^{1-h} \int_{x+h}^{1} \frac{f'(x)f(y)}{y-x} dxdy =$$

$$= \int_{-1}^{1-h} \left[\left[\ln(y-x) f'(x) f(y) \right]_{x+h}^{1} - \int_{x+h}^{1} \ln(y-x) f'(x) f'(y) dy \right] dx =$$

$$= -\int_{-1}^{1-h} \left[\ln(h) f'(x) f(x+h) - \int_{x+h}^{1} \ln(y-x) f'(x) f'(y) dy \right] dx,$$

$$G_{S_2(h)}(f) = \int_{-1+h}^{1} \int_{-1}^{x-h} \frac{f'(x)f(y)}{y-x} dxdy =$$

$$= \int_{-1+h}^{1} \left[\left[ln(x-y) f'(x) f(y) \right]_{-1}^{x-h} - \int_{-1}^{x-h} ln(x-y) f'(x) f'(y) dy \right] dx =$$

$$= \int_{-1+h}^{1} \left[ln(h) f'(x) f(x-h) - \int_{-1}^{x-h} ln(x-y) f'(x) f'(y) dy \right] dx.$$

It results that

$$\int_{-1}^{1} \int_{-1}^{1} \frac{f'(x)f(y)}{y-x} dxdy \doteqdot \lim_{h \to 0} G_{S_1(h)}(f) + G_{S_1(h)}(f) =$$
$$= \lim_{h \to 0} - \int_{-1+h}^{1} \int_{-1}^{x-h} \ln(x-y) f'(x) f'(y) dydx +$$

$$+ \ln(h) \left(\int_{-1+h}^{1} f'(x) f(x-h) dx - \int_{-1}^{1-h} f'(x) f(x+h) dx \right) +$$

$$- \int_{-1}^{1-h} \int_{x+h}^{1} \ln(y-x) f'(x) f'(y) dydx = - \int_{-1}^{1} \int_{-1}^{1} \ln|y-x| f'(x) f'(y) dxdy.$$

The thesis is obtained by observing that:

$$\left| \int_{-1}^{1} \int_{-1}^{1} \ln|y-x| f'(x) f'(y) dxdy \right| \leq \int_{-1}^{1} |f'(x)| \int_{-1}^{1} |\ln|y-x|| |f'(y)| dydx, \quad (39)$$

and that from the Hölder disequality we have:

$$\int_{-1}^{1} |\ln|y-x|| \left| f'(y) \right| dy \leq \left| \left| f' \right| \right|_{\mathscr{L}^{1+\varepsilon}(-1,1)} ||\ln|y-x|||_{\mathscr{L}^{\frac{1+\varepsilon}{\varepsilon}}(-1,1)}.$$

After having observed that, for suitable constants $\delta > 0$ and $C \in \mathbb{R}$, it results:

$$\ln|y-x| \leq \frac{C}{|y-x|^{\delta}} \quad \forall |y-x| \in]0,2],$$

we obtain: $||\ln|y-x|||_{\mathscr{L}^{\frac{\varepsilon+1}{\varepsilon}}(-1,1)} < \eta$, $\eta \in \mathbb{R}$. From (39) we have, finally:

$$\left| \int_{-1}^{1} \int_{-1}^{1} \ln|y-x| f'(x) f(y) dx dy \right| \leq \eta \, ||f'||_{\mathscr{L}^{1+\varepsilon}(-1,1)} ||f'||_{\mathscr{L}^{1}(-1,1)} < \infty \tag{40}$$

as required. □

Before proving the convexity of the functional, we recall the following:

Definition 3 *Let be K a vector space. A function $f : K \to \mathbb{R}$ is called convex, if and only if*

$$(1-\alpha) f(x) + \alpha f(y) \geq f((1-\alpha)x + \alpha y)), \quad \forall x, y \in K, \quad \forall \alpha \in [0,1]. \tag{41}$$

We say that function f is strictly convex, if and only if the inequality (41) holds strictly.

Equivalently,

Theorem 2 *Let K be a vector space and $f : K \to \mathbb{R}$ be a function whatever. f is strictly convex on K, if and only if $\forall x, y \in K$ the quotient ratio*

$$t \to R_y(t) = \frac{f(x+ty) - f(x)}{t}, \quad t \in \mathbb{R}_+ \backslash \{0\}$$

is an increasing function.

Proposition 2 *Let be:*

$$\mathscr{X} = \{f \in AC([-1,1]),\ f' \in \mathscr{L}^{1+\varepsilon},\ f(-1) = f(1) = 0\}$$

and let us define the functional

$$J : \mathscr{X} \to \mathbb{R},$$

putting

$$J(f) = \int_{-1}^{1} \int_{-1}^{1} \frac{f'(x) f(y)}{y-x} dx dy,$$

where the double integral on the right-hand side exists in the Cauchy principal. So, we have:

(a) the functional J is not strictly convex on $\mathscr{X}$;
(b) the functional J ìs strictly convex on $\mathscr{X}^+ := \{f \in \mathscr{X} : J(f) > 0\}$.

Proof. (a) We calculate the difference quotient of J for $f, g \in \mathscr{X}$ whatever:

$$R_g(t) = \frac{J(f+tg)-J(f)}{t}$$

$$= \frac{1}{t}\left(\int_{-1}^{1}\int_{-1}^{1}\frac{(f'(x)+tg'(x))(f(y)+tg(y))}{y-x}dxdy - \int_{-1}^{1}\int_{-1}^{1}\frac{f'(x)f(y)}{y-x}dxdy\right)$$

$$= \int_{-1}^{1}\int_{-1}^{1}\frac{f'(x)g(y)+g'(x)f(y)}{y-x}dxdy + t\left(\int_{-1}^{1}\int_{-1}^{1}\frac{g'(x)g(y)}{y-x}dxdy\right).$$

Now we calculate the derivative of the difference quotient:

$$R_g'(t) = \int_{-1}^{1}\int_{-1}^{1}\frac{g'(x)g(y)}{y-x}dxdy. \tag{42}$$

After Theorem 2 the functional J is not, in general, strictly convex; in fact there exist functions g for which the difference quotient $R_g(t)$ is decreasing, as for example $g = -2+\frac{3}{4}(1-x^2)^2$ because, $\forall f \in \mathscr{X}$, we get: $R_g'(t) < 0$.

(b) The strict convexity comes from theorem 2, if we adjoin, as an hypothesis for the set $\mathscr{X}$, that the condition $J(f) > 0$ holds. □

The consequence of Proposition 2 is that, if the minimum for f exists, then it is unique.

7 Appendix 2. Image Space Analysis

The study of the properties of the image of a real-valued function is an old one. However, in most cases the properties of the image have not been the purpose of the study and their investigation has occurred as an auxiliary step towards other achievements [2].

Traces of the idea of studying the images of functions involved in a constrained extremum problem go back to the work of C. Carathéodory. In the 1950s, R. Belman, with his celebrated maximum principle, proposed – for the first time in the field of Optimization – to replace the given unknown by a new one which runs in the image; however, also here the image is not the main purpose. Only in the late 1960s and 1970s some Authors, independently from each other, brought explicitly such a study into the field of Optimization (see Sect. 3.2 of [2]).

The approach consists in introducing the space, call it *Image Space* (for short, IS), where the images of functions of the given extremum problem run. Then a new problem is defined in the IS, which is equivalent to the given one. In a certain sense, such an approach has some analogies with what happens in the Theory of Measure when one goes from Mengoli-Cauchy-Riemann measure to the Lebesgue one.

The analysis in the IS must be viewed as a preliminary and auxiliary step – and not as a concurrent of the analysis in the given space – for studying a constrained extremum problem. When a statement has been achieved in the IS, then, of course, we have to write the corresponding (equivalent) statement in terms of the given space; the latter is, in general, difficult to be conceived without having at disposal the former. If this aspect is understood, then the IS Analysis may be highly fruitful. In fact, in the IS we may have a sort of "regularization": the conic extension (see Definition 5) of the image set (see Definition 4) of the given extremum problem may be convex or continuous or smooth when the given extremum problem does not enjoy the same property, so that convex or continuous or smooth analysis can be developed in the IS, but not in the given space. If the image set of an extremum problem is finite dimensional (as happens to (4)-(6)), then it can be analysed, in the IS, by means of the some mathematical concepts which are used for the finite dimensional case, even if the domain of the given problem (χ in (4)-(6)) is infinite dimensional. If the image set is infinite dimensional, by means of a suitable use of the selection theory of point-to-set maps, it is possible to postpone such an infinite dimensionality to the introduction of the IS, which, therefore, can be held finite dimensional. In this section, we understand that suitable assumptions have been made in order to let the extrema be achieved.

The IS approach arises naturally in as much as an optimality condition for an extremum problem is achieved through the impossibility of a system. By paraphrasing the very definition of global minimum point for (4)-(6), we can say that $\overline{\Gamma} \in R := \{\Gamma \in \chi : \rho V_\infty \int_T \Gamma(x)dx - c = 0\}$ is a *global minimum point*, iff the system (in the unknown Γ):

$$f_{\overline{\Gamma}}(\Gamma) := f(\overline{\Gamma}) - f(\Gamma) > 0, \quad g(\Gamma) = 0, \quad \Gamma \in \chi \tag{43}$$

is impossible. This system leads immediately to introduce the image set of (4)-(6).

Definition 4 *The set*

$$\mathscr{K}_{\overline{\Gamma}} := \{(u,v) \in \mathbb{R}^2 : u = f_{\overline{\Gamma}}(\Gamma), \quad v = g(\Gamma), \quad \Gamma \in \chi\}$$

is called the image *of* (4)-(6).

By introducing the set:

$$\mathscr{H} := \{(u,v) \in \mathbb{R}^2 : u > 0, \quad v = 0\},$$

which reflects the conditions of (43), it is trivial to state the following:

Proposition 3 $\overline{\Gamma} \in R$ *is a global minimum point of* (4)-(6)*, if and only if:*

$$\mathscr{H} \cap \mathscr{K}_{\overline{\Gamma}} = \emptyset. \tag{44}$$

In passing, it is worth noting that minimization is the way of reading (in the sense of Galilei) the laws of nature (or human behaviour), while the mathematical core of an extremum problem consists in proving the impossibility of a system or the disjunction of two sets, as (43) and (44) show.

As announced, we can now introduce the *image problem*:

$$\max(u), \quad \text{s.t. } (u,v) \in \mathscr{K}_{\overline{\Gamma}}, \quad v = 0, \tag{45}$$

and prove the following:

Proposition 4 *Problems* (4)-(6) *and* (45) *are equivalent, in the sense that* $(\widehat{u}, \widehat{v})$ *is a global maximum point of* (45)*, if and only if it is the image, through the map* $(f_{\overline{\Gamma}}(\Gamma), g(\Gamma))$*, of a global minimum point, say* $\widehat{\Gamma}$*, of* (4)-(6)*, and we have:*

$$f(\overline{\Gamma}) - \widehat{u} = f(\widehat{\Gamma}). \tag{46}$$

Proof. Only if. $(\widehat{u}, \widehat{v}) \in \mathscr{K}_{\overline{\Gamma}} \cap (\mathbb{R} \times \mathbb{O}) \Rightarrow \exists \widehat{\Gamma} \in \chi$, such that:

$$u = f(\overline{\Gamma}) - f(\Gamma), v = g(\Gamma) = 0.$$

Taking into account these relations (the first of which proves the last claim), the assumption:

$$\widehat{u} \geqslant u, \quad \forall (u,v) \in \mathscr{K}_{\overline{\Gamma}} \cap (\mathbb{R} \times \mathbb{O}),$$

implies $f(\overline{\Gamma}) - f(\widehat{\Gamma}) \geqslant f(\overline{\Gamma}) - f(\Gamma)$ or $f(\widehat{\Gamma}) \leqslant f(\Gamma), \forall \Gamma \in \chi$.
If. Set $\widehat{u} := f(\overline{\Gamma}) - f(\widehat{\Gamma}), \widehat{v} := g(\widehat{\Gamma})$, so that:

$$(\widehat{u}, \widehat{v}) \in \mathscr{K}_{\overline{\Gamma}} \cap (\mathbb{R} \times \mathbb{O}).$$

From the assumption we draw $f(\widehat{\Gamma}) \leqslant f(\Gamma), \forall \Gamma \in R$; by setting:
$u := f(\overline{\Gamma}) - f(\Gamma)$ and $v := g(\Gamma)$, we have $f(\overline{\Gamma}) - f(\widehat{\Gamma}) \geqslant f(\overline{\Gamma}) - f(\Gamma), \forall \Gamma \in R$, and hence $\widehat{u} \geqslant u, \forall (u,v) \in \mathscr{K}_{\overline{\Gamma}} \cap (\mathbb{R} \times \mathbb{O})$. □

Note that, while (4)-(6)is infinite dimensional (its unknown runs in a Banach space), (45) is finite dimensional (its unknown runs in the Euclidean plane).

The theory of constrained extrema is full of proposals for changing the data of the given problem, without losing the extremum and extremum points, and with the purpose of adding a desired property to the problem. Such proposals have been made essentially with reference to the given space. The IS approach suggests a new proposal, based on the following definition. *cl* denotes closure, and the difference is in the vector sense.

Definition 5 *Let* $\mathscr{Z} \subset \mathbb{R}^2$ *denote a generic set of the IS associated with* (4)*.* $\mathscr{E}$ *will denote the map which sends* $\mathscr{Z}$ *into* $\mathscr{Z} - cl\mathscr{H} \subset \mathbb{R}^2$*; it is called* conic extension *of* $\mathscr{Z}$*.*

Of course $\mathscr{Z} \subseteq \mathscr{E}(\mathscr{Z})$. In the sequel, we will consider only the conic extension of the image set, or $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$. The above definition has been given for the particular problem (4)-(6); obviously, it can be given for a general extremum problem (see [2], Def. 3.2).

Proposition 5 (44) *holds, if and only if:*

$$\mathscr{H} \cap \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) = \emptyset \tag{47}$$

Proof. If. It is an obvious consequence of the inclusion $\mathscr{K}_{\overline{\Gamma}} \subseteq \mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$.
Only if. Ab absurdo, suppose that $\exists z^1 \in \mathscr{K}_{\overline{\Gamma}}, \exists z^2 \in cl\mathscr{H}$ (so that $z^1 - z^2 \in \mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$), and that $z^1 - z^2 \in \mathscr{H}$. Then, being $\mathscr{H}$ the positive u-semiaxis (of the IS), we have:

$$z^1 = (z^1 - z^2) + z^2 \in \mathscr{H} + cl\mathscr{H} = \mathscr{H},$$

and hence (44) is contradicted. □

The above proposition shows that the optimality condition (44) still holds, if the image set (and therefore the data of (4)-(6)) are modified according to Definition 5. This has an obvious consequence on the image problem (45), as shown by the following:

Proposition 6 *Let Condition 1 hold.*

(i) *Problems* (45) *and*

$$\max(u), \quad s.t. \quad (u,v) \in \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}), \quad v = 0, \tag{48}$$

are equivalent in the sense of having the same maximum and maximum points.

(ii) *Problem* (48) *has maximum.*

Proof.

(i) Straightforward consequence of Propositions 3-5.

(ii) Because of Condition 1, $-f_{\overline{\Gamma}}(\Gamma)$ (see (43)) is coercive. Because of Proposition 2(b), $f_{\overline{\Gamma}}(\Gamma)$ is strictly concave. $g(\Gamma)$ is linear. Therefore, in the IS, the projection of $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ – as well as of $\mathscr{K}_{\overline{\Gamma}}$ – on the u-semiaxis is a closed (and bounded) segment. Hence, the assumptions of Theorem 3.2.3 of [2] are fulfilled. Such a theorem can thus applied to achieve the thesis. □

As announced in Sect. 3 (just before Condition 1), it is possible to prove the existence of the minimum in (4)-(6) through IS: this is done by the above Proposition 6.

In section 7, we have shown that a feasible $\overline{\Gamma} \in R$ is a (global) minimum point of (4)-(6), iff (44) holds. In the general case (but also in the present one), to prove (44) is a difficult task. Therefore, a way of overcoming such a drawback consists in trying to show that $\mathscr{H}$ and $\mathscr{K}_{\overline{\Gamma}}$ lie in two disjoint sets. The separation theory, whose "root"is the Hahn–Banach Linear Extension Theorem (but it was already present, even if in an implicit form, in Euclid!), is of great help.

Let us consider the function $w : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$w(u,v;\theta,\lambda) = \theta u + \lambda v, \quad \theta,\lambda \in \mathbb{R} \tag{49}$$

For each pair $(\theta,\lambda) \neq 0, \quad w(u,v;\theta,\lambda) = 0$ indetifies obviously a line, say H^0, through the origin, of the IS (i.e., $\mathbb{R}^2$), where (u,v) runs; the IS is then split into two disjoint halfplanes:

$$H^-(\theta,\lambda,k) := \{(u,v) \in \mathbb{R}^2 : \theta u + \lambda v \leqslant k\}, \quad \theta,\lambda,k \in \mathbb{R},$$
$$H^+(\theta,\lambda,k) := \{(u,v) \in \mathbb{R}^2 : \theta u + \lambda v > k\}, \quad \theta,\lambda,k \in \mathbb{R}.$$

Of course, we have $\mathscr{H} \subset H^+(\theta,\lambda,0)$, iff $\theta \neq 0$; thus, under this assumption, in order to prove (44) (and, hence, the optimality of $\overline{\Gamma}$) it is sufficient to show that $\exists \theta,\lambda \in \mathbb{R}$, with $\theta \neq 0$, such that:

$$\mathscr{K}_{\overline{\Gamma}} \subseteq H^-(\theta,\lambda,0), \tag{50}$$

or, equivalently (Proposition 5),

$$\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \subseteq H^-(\theta,\lambda,0). \tag{51}$$

In the general case, (50) – or (51) – is not necessary, as trivial examples show. However, it will be shown that, in the present case (4)-(6), the inclusion (50) – or (51) – is also necessary.

Let ∂S and $card\,S$ denote the boundary and the cardinality of the set S, respectively; the difference between sets is denoted by "\"; $H^0(\theta,\lambda,k) := \{(u,v) \in \mathbb{R}^2 : \theta u + \lambda v = k\}$ denotes a line (iff $(\theta,\lambda) \neq 0$) of the IS.

Proposition 7 *Let $\overline{\Gamma} \in \chi$. $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ enjoys the following properties:*

(i) *$\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ is strictly convex;*
(ii) *$\partial\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \subset \mathscr{K}_{\overline{\Gamma}}$;*
(iii) *$\forall (u,v) \in \partial\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$, $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ admits a support, or $\exists(\theta,\lambda) \in \mathbb{R}^2$, with $\theta \neq 0$, and $\exists k \in \mathbb{R}$, such that:*

$$\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \subset H^-(\theta,\lambda,k), \quad S := \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \cap H^0(\theta,\lambda,k) \neq \emptyset, \quad card\,S = 1; \tag{52}$$

the same happens to $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$, or:

$$\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \subset H^-(\theta,\lambda,k), \quad \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \cap H^0(\theta,\lambda,k) = S; \tag{53}$$

(iv) *$\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ is* regular*, in the sense that, $\forall (u,v) \in \partial\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$, no supporting line at (u,v) to $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ (boundary of the Bouligand tangent cone to $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$, is parallel to the u–axis of the IS.*
(v) *at $v \geqslant 0$, a supporting line $H^0(\theta,\lambda,k)$ sub (iii) has $\theta > 0$ and $\lambda > 0$.*

Proof. (i) Because of Proposition 2, $f(\Gamma)$ is strictly convex, so that $f_{\overline{\Gamma}}(\Gamma)$ (see (43)) is strictly concave. Hence, taking into account that $g(\Gamma)$ is linear in Γ and that $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ is the hypograph of $\mathscr{K}_{\overline{\Gamma}}$ (when $\mathscr{K}_{\overline{\Gamma}}$ is viewed as the point–to–set maps $v \rightrightarrows u$), $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ turns out to be strictly convex. **(ii)** It is a consequence of (i) and of the fact that $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ is the hypograph of $\mathscr{K}_{\overline{\Gamma}}$. **(iii)** The first two conditions of (52) come from the convexity of $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ and, as it concerns the existence of $\theta \neq 0$, from (iv); the last part of (52) is a consequence of the strict convexity of $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$. Passing to (53), it is enough to observe that $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ is the hypograph (in the sense specificed sub (i)) of $\mathscr{K}_{\overline{\Gamma}}$. **(iv)** Because of the assumption 1 (see the end of introduction to Sect. 3), $\forall v > 0, \exists (u,v) \in \mathscr{K}_{\overline{\Gamma}}$ (more general condition than the so-called Slater constraint qualification). Therefore, the existence of a supporting line $H^0(\theta,\lambda,k)$, parallel to the u–axis, account taken of (i), would require the boundedness of $\mathscr{K}_{\overline{\Gamma}}$ with respect to v and contradict the assumption. **(v)** It is a consequence of (iv) and of the assumption 1. □

We are now ready to show that (50) and (51) are also sufficient.

Proposition 8 *$\overline{\Gamma} \in R$ is a (global) minimum point of* (4)-(6)*, if and only if* $\exists (\overline{\theta},\overline{\lambda}) > 0$*, such that:*

$$\mathscr{K}_{\overline{\Gamma}} \subset H^-(\overline{\theta},\overline{\lambda},0) \quad \textit{or equivalently} \quad \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \subset H^-(\overline{\theta},\overline{\lambda},0). \tag{54}$$

Proof. The sufficiency is an obvious consequence of what has been noted about (50). With regard to the necessity, the assumption that $\overline{\Gamma}$ be a (global) minimum point of (4)-(6) implies (44) or (47). Because of Proposition 7(i), $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$ and $\mathscr{H}$ are separable; because of Proposition 7(iv), the separation line is of type $H^0(\overline{\theta},\overline{\lambda},0)$ and does not contain the u–axis; because of Proposition 7(v), $\overline{\theta}$ and $\overline{\lambda}$ are positive. Thus, the latter of (54) follows; the former is a consequence of the inclusion $\mathscr{K}_{\overline{\Gamma}} \subset \mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$. □

Since it is not easy to verify (54), it comes spontaneous to try to express the inclusion (54) through some extremum operators. In general, we cannot have equivalence; here it happens, due to Proposition 7.

Proposition 9 *Let $\overline{\Gamma} \in R$.*
(i) *The equality*

$$\min_{\theta,\lambda>0} \max_{(u,v)\in\mathscr{K}_{\overline{\Gamma}}} (\theta u + \lambda v) = 0 \quad \textit{or} \quad \min_{\theta,\lambda>0} \max_{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})} (\theta u + \lambda v) = 0 \tag{55}$$

are equivalent, respectively, to (54)*.*
(ii) *$\overline{\Gamma}$ is a global minimum point of* (4)-(6)*, if and only if* (55) *hold.*

Proof. (i) Taking into account the definition of $\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})$, it is enough to prove the equivalence between the second of (54) and the second of (55). Let the second of (54) hold. Because of Proposition 7(iii)–(v), $\exists \overline{\theta},\overline{\lambda} > 0$ and $\exists (\overline{u},\overline{v}) \in \mathbb{R}^2$, such that:

$$\overline{\theta} u + \overline{\lambda} v \leqslant 0, \quad \forall (u,v) \in \mathscr{E}(\mathscr{K}_{\overline{\Gamma}}), \quad \theta u + \lambda v = 0 \Leftrightarrow (u,v) = (\overline{u},\overline{v}).$$

Then, the maximum in the second of (55) is a non–negative function of (θ,λ), which takes the value zero. Hence, the second of (55) follows. The reverse implication is obvious. **(ii)** Straightforward consequence of (i). □

The left-hand side of the first of (55) is called *image dual problem*. In looking at the problems in (55), a question comes spontaneous: what kind of problem will we find, if the operators will be exchanged each other? Surprisingly, we do not find any problem, but just (45); this is expressed by the following:

Theorem 3 (Image Duality) *Let $\overline{\Gamma} \in R$. We have:*

$$\min_{\theta,\lambda>0} \max_{(u,v)\in\mathscr{K}_{\overline{\Gamma}}} (\theta u + \lambda v) = \max_{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})} \min_{\theta,\lambda>0} (\theta u + \lambda v) = \max_{\substack{(u,v)\in\mathscr{K}_{\overline{\Gamma}} \\ v=0}} (u), \quad (56)$$

or:

$$\min_{\theta,\lambda>0} \max_{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})} (\theta u + \lambda v) = \max_{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}})} \min_{\theta,\lambda>0} (\theta u + \lambda v) = \max_{\substack{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \\ v=0}} (u). \quad (57)$$

Proof. If $v \neq 0$, then the minimum in the second side of (56) may not exist and, in its place, the infimum – which is a function of (u,v) – is less than the value it takes at $v = 0$. Therefore, it is not restrictive to add the constraint $v = 0$ to the minimum in the second side of (56). Hence, taking into account that, due to the homogeneity of $\theta u + \lambda v$, it is not restrictive to assume $\theta = 1$, so that the minimization becomes obvious, the second equality of (56) follows. Between the first and second sides of (56) the inequality $\geqslant$ holds as a special case of a well known and classic inequality. The equality is a consequence of Proposition 7. A quite similar reasoning proves (57). □

Once the IS Analysis related to a given problem has been accomplished and some (image) statements have been proved in the IS, then such statements must be transferred to the given space, finding what we can call counterimage statements. To find image statements is, in general, much easier than to search for them directly in the given space; sometimes, in the given space it is difficult even to conceive a statement of this type. This is the main role of the IS Analysis.

Now, let us write the counterimage statements of Proposition 8 and Theorem 3. To this end, consider the function:

$$\mathscr{L}(\Gamma;\theta,\lambda) := \theta f(\Gamma) - \lambda g(\Gamma) = \int_T \int_T [\theta \frac{\rho}{4\pi} \frac{\Gamma'(x)\Gamma(y)}{y-x} - \lambda \rho V_\infty \frac{\Gamma(x)}{2}] dx dy, \quad (58)$$

which is called Lagrangian function associated to (4)-(6). It expresses a (linear) combination of two entities, induced drag and the (difference between the) total lift (and a given constant, i.e. c). In the format (4)-(6), the former is considered as an objective and the latter as a constraint. To adopt such a format is subjective. Therefore, why not to consider the *reciprocal problem*:

$$\max[c + g(\Gamma)], \quad \text{subject to} \quad f(\Gamma) = d, \quad \Gamma \in \chi, \quad (59)$$

where d is a constant? In passing, we recall that the introduction of the reciprocal problem goes back to the ancient Greeks; under very general conditions (see [2], Sect. 5.5), it holds that, for suitable values of the constants c and d, a same Γ solves both (4)-(6) and (59); this is known as *reciprocity principle*.

We will see that the theory of duality offers a way for overcoming the embarrassment of being obliged to choose between the formats (4)-(6) and (59).

Proposition 10 *$\overline{\Gamma} \in \chi$ is a (global) minimum point of* (4)-(6)*, if and only if $\exists(\overline{\theta},\overline{\lambda}) > 0$, such that:*

$$\mathscr{L}(\overline{\Gamma};\theta,\lambda) = \mathscr{L}(\overline{\Gamma};\overline{\theta},\overline{\lambda}) \leqslant \mathscr{L}(\Gamma;\overline{\theta},\overline{\lambda}), \quad \forall \Gamma \in \chi, \forall \theta,\lambda > 0. \tag{60}$$

Proof. The equality in (60) holds, iff $\overline{\Gamma} \in R$ or iff $\overline{\Gamma}$ is feasible for (4)-(6). When such an eqaulity holds, the inequality in (60) is equivalent to the optimality of $\overline{\Gamma}$, due to Proposition 8 (first inclusion). □

Note that, unlike Proposition 8, in Proposition 10 $\overline{\Gamma}$ is assumed to merely belong to χ. A triplet $(\overline{\Gamma},\overline{\theta},\overline{\lambda})$ fulfilling (60) is called *saddle point* of $\mathscr{L}$; the second side of (60) is the corresponding *saddle value*.

Theorem 4 (Duality) *Let $\overline{\Gamma} \in R$. We have:*

$$\max_{\theta,\lambda>0} \; \min_{\Gamma\in\chi} \mathscr{L}(\Gamma;\theta,\lambda) = \min_{\Gamma\in\chi} \; \max_{\theta,\lambda>0} \mathscr{L}(\Gamma;\theta,\lambda) = \min_{\Gamma\in R} f(\Gamma). \tag{61}$$

Proof. It is enough to use (43), Definition 4 and (58), and replace u and v in (56) with their expression in terms f, g, χ, and, finally apply Theorem 3. □

The first side of (61) is called the *dual problem* of (4)-(6), or of the third side of (61). The second side on (61) has been obtained from the dual problem, by exchanging the order of the extremum operators; as announced above, it equals the primal problem. Note that the dual problem has nothing to share with the reciprocal problem, as it is easy to see, by comparing the first side of (61) with (59).

The IS Analysis allows one to achieve several other important informations. Proposition 7 – and, in particular, its (iii) – suggests the introduction of the following function:

$$u(\xi) := \max_{\substack{(u,v)\in\mathscr{K}_{\overline{\Gamma}} \\ v=\xi}} (u) = \max_{\substack{(u,v)\in\mathscr{E}(\mathscr{K}_{\overline{\Gamma}}) \\ v=\xi}} (u), \tag{62}$$

the second equality in (62) being due to Proposition 7(i). The function (62) is called *perturbation function* associated with (4)-(6). When a problem does not enjoy a convexity property like (i) of Proposition 7, then the definition of the perturbation function is more general than (62). The perturbation function gives the value of the image problem (45) or (48), when the constraint $v = 0$ is replaced by $v = \xi$; since the image problem is related to (4)-(6) by a relationship of type (46), then $u(\xi)$ allows one to know the change in the minimum in (4)-(6) consequent to a change in the right-hand side of (5), where now zero is replaced by ξ. Furthermore, the (sub)derivative of $u(\xi)$ gives the (instantaneous) velocity of the minimum of (4)-(6) with respect to the right-hand side of (5), namely ξ. Inparticular, at $\xi = 0$, the (instantaneous) velocity of the minimum of (4)-(6) with respect to the right-hand side of (5) is given by $\overline{\lambda}/\overline{\theta}$. This number, which is nothing more than the classic Lagrangian multiplier, allows one *to evaluate the change in the minimum induced drag consequent to a change in the value at which the total lift is constrained*.