Free Vibration Analysis of Laminated Composite Plates with Arbitrary Shape

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Abstract

In this paper, the undamped free vibration analysis of arbitrary shaped laminated composite plates with general stacking sequences is conducted based on the first-order shear deformation theory. The finite element method is used to obtain the plate's vibrational characteristics by introducing a six-nodded triangular element, i.e., natural frequencies and the corresponding mode shapes. The element considered is a higher-order triangular element. Each node includes five degrees of freedom. Gaussian numerical integration is used to calculate the mass and stiffness matrices. The whole solution method is implemented within the MATLAB. The convergence of the results has been investigated, and results have been compared against some available data in the literature and also commercial software ANSYS in which three-dimensional analysis is used. Excellent agreements have been observed. The effects of several parameters – such as boundary conditions, geometry, and lay-ups – on the natural frequencies are studied in detail.

Keywords: Free Vibration; Arbitrary Shape, Laminated Composite; Plate; Finite Element.

1. Introduction

Composite structures have been widely used in diverse engineering fields due to their outstanding characteristics such as light-weight, high-strength, and shock resistance. In most practical cases, plates do not have regular geometries – like squares, rectangles, triangles, etc. – and feature entirely arbitrary and complex shapes.

A fundamental contribution to the vibration of plates with regular shapes dates back to research done by Leissa [1]. Wang et al. [2] carried out the vibrational analysis of a rectangular plate. Shi et al. [3] applied the Galerkin method to the vibrational analysis of a rectangular plate with fully clamped boundary conditions. Abedi et al. [4] investigated the vibration analysis of a rectangular composite plate with general stacking sequences and edge restraints using the Ritz method combined with Lagrange multipliers. Vibration analysis of symmetric trapezoidal plates based on FSDT was conducted by Zamani et al. [5]. Nallim and Oller [6] proposed an analytical-numerical approach for the dynamic analysis of laminated composite plates with arbitrary quadrilateral geometry. Chen et al. [7] used the finite element method (FEM) for the free vibration analysis of arbitrary quadrilateral plates with elastic edge supports. Carrera [8-9] introduced a unified formulation for the finite element analysis of multi-layered plates, thus paving the way to analyze plates with general geometries.

In this paper, the undamped free vibration of arbitrary shaped laminated composite plates is studied. First-order Shear Deformation Theory (FSDT) is used, and the Finite Element Method (FEM) is adopted to discretize and solve the posed problems numerically. In particular, the arbitrary in-plane plate geometries are approximated through six-noded triangular elements. Free vibration analysis is conducted based on a generalized eigenvalue problem. To the best of the authors knowledge, it is the first time that undamped free vibration of arbitrary shaped laminated composite plates with the aid of FEM and FSDT by introducing higher order triangular elements is considered. The whole solution method is implemented within the MATLAB software code.

The paper is arranged as follows: a brief description of the problem is explained in Section 2. Section 3 presents a mathematical formulation for laminate composite plates considering FSDT. Several numerical examples are provided in Section6. Finally, we close our paper with some concluding remarks.

2. Problem Description

Let us consider a multi-layered composite plate with arbitrary lay-up and arbitrary in-plane shape represented by region A (Figure 1). The plate is made up of n orthotropic layers with total thickness h. We fix a Cartesian right-handed coordinate system Oxyz with the origin O, x- and y-axes on the plate mid-plane.

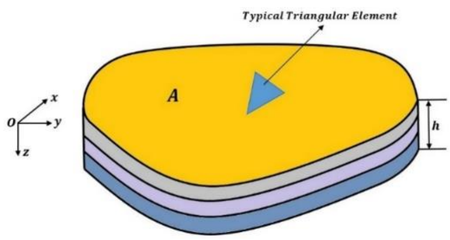


Figure 1. Laminated composite plate with arbitrary lay-up and in-plane shape

3. Mathematical Formulation

3.1 Stress Resultants

According to the FSDT assumptions, the three-dimensional displacement field in the plate can be represented in terms of components along the x-, y-, and z-directions, respectively, as follows:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\,\psi_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\,\psi_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{1}$$

In Eq. (1), u_0, v_0, and w_0 are the mid-plane displacements, t denotes time, ψ_x and ψ_y are the angles of rotation of the normal to the mid-plane about the y- and x-axes, respectively.

Using the strain-displacement relations of linear elasticity, the strain components in compact matrix form are:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} \\ v_{0,x} \\ u_{0,y} + v_{0,x} \end{Bmatrix} + z \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{2}$$

Besides, the transverse shear strains γ_{xz} and γ_{yz} can be written as:

$$\begin{aligned} \gamma_{xz} &= u_{,z} + w_{,x} = \psi_x + w_{0,x} \\ \gamma_{yz} &= v_{,z} + w_{,y} = \psi_y + w_{0,y} \end{aligned} \tag{3}$$

The stress-strain relations for the kth layer in matrix form are [10]:

$$\begin{gathered} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)} \\ \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \end{gathered} \tag{4}$$

where $\bar{Q}_{ij}$ are the transformed reduced stiffness constants [10]. The stress resultants turn out to be:

$$\begin{gathered} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \\ \begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \end{gathered} \tag{5}$$

where $A_{ij}, B_{ij}$and D_{ij} are the extension stiffness, extension-bending coupling stiffness, and bending stiffness, respectively; these can be calculated as:

$$\begin{aligned} [A_{ij}, B_{ij}, D_{ij}] &= \sum_{k=1}^{N} \left(\bar{Q}_{ij}\right)_k \int_{z_{k-1}}^{z_k} [1, z, z^2] dz \qquad i,j = 1,2,6 \\ A_{ij} &= k_{ij} \sum_{k=1}^{N} \left(\bar{Q}_{ij}\right)_k \int_{z_{k-1}}^{z_k} [1, z, z^2] dz \qquad i,j = 4,5 \end{aligned} \tag{6}$$

In Equation (6), k_{ij} are the shear correction factors, here for simplicity considered to be $5/6$, like for homogenous beams of rectangular cross-section.

3.2 Potential and Kinetic Energies of the Plate

The strain potential energy of the plate based on FSDT is:

$$U_p = \frac{1}{2}\iint_A \left(N_x\varepsilon_x^0 + N_y\varepsilon_y^0 + N_{xy}\gamma_{xy}^0 + M_xk_x + M_yk_y + M_{xy}k_{xy} + Q_{yz}\gamma_{yz} + Q_{xz}\gamma_{xz}\right)dxdy \tag{7}$$

Likewise, the kinetic energy of the multi-layered plate is [10]:

$$T = \frac{1}{2}\iint_A \left[I_0\left(u_{,t}^2 + v_{,t}^2 + w_{,t}^2\right) + 2I_1\left(u_{,t}\psi_{x,t} + v_{,t}\psi_{y,t}\right) + I_2(\psi_{x,t}^2 + \psi_{y,t}^2)\right]dxdy \tag{8}$$

Where

$$(I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)(1, z, z^2)dz \tag{9}$$

are the inertia moments of the plate, and ρ is the material density (possibly, varying from layer to layer).

4. Finite Element Method

In this study, triangular elements are used to discretize the plate and approximate its arbitrary in-plane geometry. The element considered is a higher-order triangular element with six nodes. Each node includes five degrees of freedom $(u_0, v_0, w_0, \psi_x, \psi_y)$, and thus each element totally has 30 degrees of freedom.

Lagrange interpolation functions are used to describe the generalized displacement fields inside the element [11]:

$$\mathbf{u}(x, y) = \sum_{i=1}^{6} N_i\,\mathbf{u_i} \tag{10}$$

where $N_1, N_2, N_3, N_4, N_5 N_6$ are shape functions, that will be calculated in the next section. Equation (10) can be rewritten as follows:

$$\begin{aligned} u_0(x, y) &= [N_u]\{d\} \\ v_0(x, y) &= [N_v]\{d\} \\ w_0(x, y) &= [N_w]\{d\} \\ \psi_x(x, y) &= \left[N_{\psi_x}\right]\{d\} \\ \psi_y(x, y) &= \left[N_{\psi_y}\right]\{d\} \end{aligned} \tag{11}$$

Where

$$\{d\} = \left\{u_1, v_1, w_1, \psi_{x_1}, \psi_{y_1} \cdots, u_6, v_6, w_6, \psi_{x_6}, \psi_{y_6}\right\}^T \tag{12}$$

is the 30×1vector of element displacements, and T denotes transpose.

Besides, $[N_u], [N_v], [N_w], \left[N_{\psi_x}\right]$ and $\left[N_{\psi_y}\right]$ are matrices of shape functions of order 1×30, and for the triangular element can be considered as [11].

By substituting the generalized displacements (11) into equations(7-10), stiffness and mass matrices and force vector can be calculated. By assembling the matrices and vectors for all of the elements in global coordinates, the mass and stiffness matrices of the whole plate together with the total force vector are calculated. Finally, the equations of motion of the discrete plate model in matrix form are written as:

$$[M]\{\ddot{\triangle}\} + [K]\{\triangle\} = \{F(t)\} \tag{13}$$

where $\{\triangle\}$ is the vector of degrees of freedom of the whole plate, $[M]$ and $[K]$ are the global mass and stiffness matrices, respectively, and $\{F(t)\}$ is the total force vector. It is noteworthy that the well-known penalty approach is used to apply the boundary conditions [11].

4.1 Shape Functions

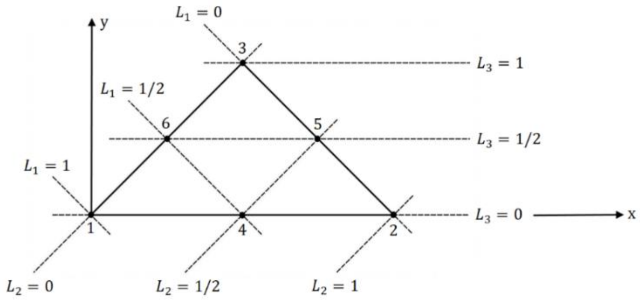


Figure 2. Area coordinates for obtaining the shape functions [11]

The shape functions for the triangular element can be expressed as [11]:

$$N_i = (2L_i - 1)L_i \qquad (i = 1, 2, 3)$$
$$N_4 = 4L_1L_2, \quad N_5 = 4L_2L_3, \quad N_6 = 4L_3L_1 \tag{14}$$

Where L_1, L_2 and $L_3$are the area coordinates (Figure 2) [10].

5. solution

By ignoring the right-hand side of equation (13) and assuming a general solution of the form $\{\triangle\} = \{\triangle_0\}e^{\hat{\imath}\omega t}$, where $\hat{\imath}$ is the imaginary unit and ω is the natural angular frequency, equation (13) yields the following generalized eigenvalue problem:

$$([K] - \omega^2[M])\{\triangle_0\} = 0 \tag{15}$$

where ω^2 is the eigenvalue, and $\{\triangle_0\}$ is the eigenvector, describing the corresponding mode shape. It is noteworthy that the well-known penalty approach is used to apply the boundary conditions [12], and the whole solution method is implemented within the MATLAB software code.

6. Results

Numerical results are presented in this section. For each case, the convergence of the response is investigated, and then they are presented. Unless mentioned otherwise, the physical properties of each layer are taken to be:

$$E_2 = 9.65\ \text{GPa}, \quad E_1 = 40E_2, \quad \nu_{12} = 0.25$$
$$G_{12} = G_{13} = 0.6E_2, \quad G_{23} = 0.5E_2, \quad \rho = 1389.23\ \text{kg/m}^3 \tag{16}$$

All layers are of equal thickness and have the same physical properties.

6.1 Validating the results

Consider the equilateral triangular plate; the dimensionless fundamental frequencies of the plate are presented in Table 1 for different lay-ups and calculation methods.

Table 1. Dimensionless frequencies ($\overline{\Omega}$) of the equilateral triangular composite plate with fully clamped boundary conditions

Lay-ups	*h/a*	[5]	ANSYS [5]	Present
[0°/90°/90°/0°]	0.05	68.17	70.61	68.180
	0.1	43.32	44.44	43.421
	0.15	31.17	31.78	31.248
[30°/60°/60°/30°]	0.05	59.18	59.28	59.210
	0.1	38.39	39.07	38.404
	0.15	29.18	29.02	28.355
[45°/60°/60°/45°]	0.05	57.13	56.21	57.18
	0.1	37.12	37.60	37.233
	0.15	27.49	28.15	27.573
[30°/90°/90°/30°]	0.05	64.54	65.16	64.736
	0.1	41.61	42.07	41.719
	0.15	30.20	30.62	30.368

The following material properties are used in this example:

$$E_1 = 40E_2,\ \ G = 0.6E_2\ ,\ \ \nu = 0.25,\ \ \ \rho = 2500\ \mathrm{kg/m^3} \tag{17}$$

The dimensionless frequencies are computed as $\overline{\Omega} = \frac{\omega a^2\sqrt{\rho/E_2}}{h}$. As can be seen, very good agreement is obtained between compared methods.

6.2 Gear-shaped Plate

Figure 3 shows a plate with the shape of a gear. The first three dimensionless frequencies of this plate with fully clamped edges for different values of *h/a* and a = 2b are presented in Table 2 where a is the length, b is the width, and h denotes the thickness of the plate. The maximum and minimum length sizes of the elements are assumed to be 0.05 mm and 0.025 mm, respectively. Also, the mentioned plate is simulated in Ansys software, and the results of the method used in the present work have been compared with the results of the same model in Ansys software. As can be seen, by increasing the ratio *h/a*, the dimensionless frequencies decrease. The corresponding mode shapes for *h/a*= 0.1 are shown in Figure 4.

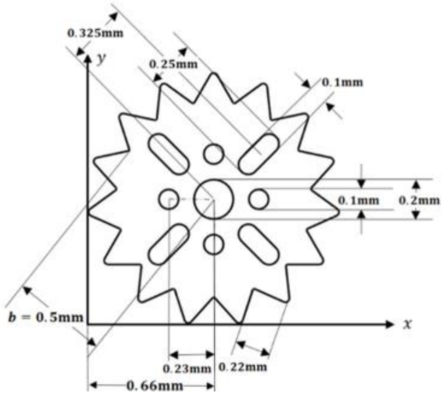
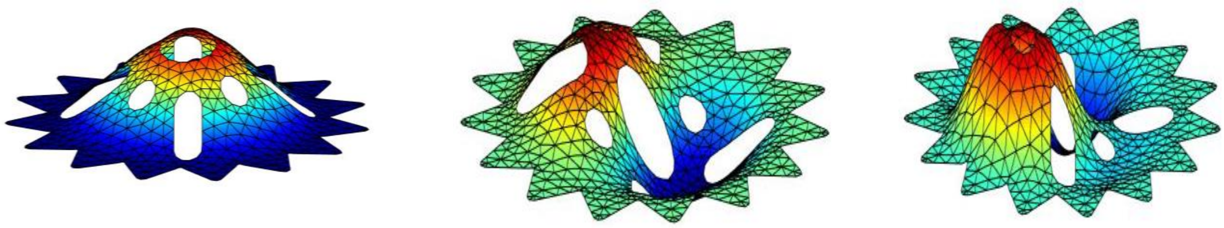


Figure 3. Gear-shaped plate

Table 2. Dimensional frequencies (Ω) of a gear-shaped plate with lay-up [0°/45°] and clamped edges

h/a	Present			ANSYS		
	Mode No.			Mode No.		
	1	2	3	1	2	3
0.1	18.2875	26.0498	31.9411	16.8836	24.6814	32.9586
0.15	14.8059	20.9229	24.3503	13.2820	18.7667	24.1895
0.2	12.4036	17.4086	19.5757	10.7415	14.9512	18.9002



1st mode 2nd mode 3rd mode

Figure 4. First three mode shapes of the gear-shaped plate with lay-up [0°/45°] and clamped edges (*h/a*=0.1)

6.3 Car Door-shaped Plate

In order to consider an arbitrary shape plate, a composite laminated plate is considered with the shape of a car door and the dimensions shown in Figure 5.

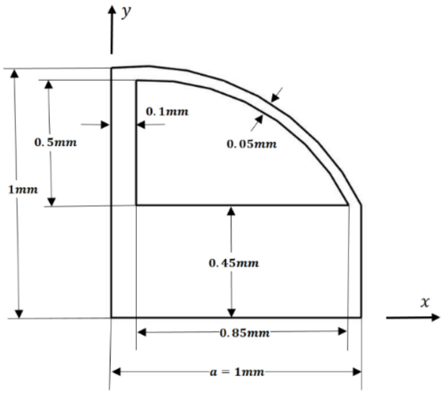
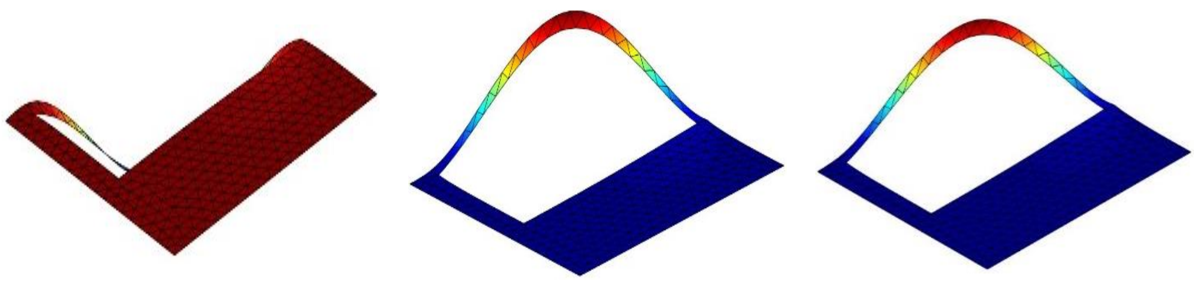


Figure 5. Laminated composite plate in the shape of a car door

The first five dimensionless frequencies (Ω) of the plate for different boundary conditions are presented in Table 3. The letters denoting the boundary conditions orderly refer to the sides at $x = 0, y = 0, x = a$ and the outer curve. In this table, each letter stands for different boundary conditions, "C" stands for Clamped, "S" means Simply Support condition, and finally, "F" is the abbreviation of Free boundary condition. The maximum length size of the elements is assumed to be 0.05 mm, and the minimum length size of the elements is considered 0.025 mm. Generally, for specified thickness and boundary conditions, the plate with lay-up [0°/90°/90°/0°] has the maximum frequency, while the one with lay-up [45°/60°/60°/45°] has the minimum frequency. The first five corresponding mode shapes for one of the considered cases are displayed in Figure 6.

Table 3. First five dimensionless frequencies (Ω) of the plate in the form of a car door with various lay-ups, boundary conditions, and thicknesses

BCs	Lay-up	h/a	Mode Number 1	2	3	4	5
CCCC	[0°/90°/90°/0°]	0.05	31.8522	66.6748	70.4839	93.0312	107.8909
		0.1	22.1188	40.6470	49.6290	60.6853	62.2869
	[30°/60°/60°/30°]	0.05	24.0237	45.9986	69.5385	70.6341	93.3291
		0.1	18.5448	32.5856	47.5445	48.3800	61.3373
	[45°/60°/60°/45°]	0.05	22.5922	38.2056	59.2007	77.3155	84.7493
		0.1	17.9163	28.8338	42.9690	50.3017	51.0488
SSSS	[0°/90°/90°/0°]	0.05	16.6767	43.3368	50.3027	53.8591	54.6942
		0.1	13.9130	14.4720	30.0697	37.0760	41.7125
	[30°/60°/60°/30°]	0.05	12.1139	31.3524	42.9285	46.0884	56.4930
		0.1	10.1377	19.7465	25.1461	31.3974	35.0092
	[45°/60°/60°/45°]	0.05	10.3445	25.2259	39.3900	41.7117	519298
		0.1	8.6396	18.8348	20.8200	30.1982	33.6242
CCFF	[0°/90°/90°/0°]	0.05	8.9665	14.7646	26.0409	26.5206	31.7833
		0.1	7.8548	12.7251	13.2603	15.8917	20.0438
	[30°/60°/60°/30°]	0.05	6.0774	13.6512	17.0998	18.3592	19.4969
		0.1	5.5149	8.5499	9.7485	11.9972	15.7514
	[45°/60°/60°/45°]	0.05	5.4004	13.6907	15.3657	17.8348	18.9504
		0.1	4.8984	7.6829	8.9174	12.0519	15.7687



1st mode   2nd mode   3rd mode

4th mode 5th mode

Figure 6. First five mode shapes of a composite car door-shaped plate with lay-up $[45°/60°/60°/45°]$ and CCFF boundary conditions (h/a=0.1)

7. Conclusion

In this paper, free vibration analysis of the composite plate with arbitrary geometry was performed using a high-order triangular element based on first-order shear deformation theory. Two fully arbitrary plates have been considered and analyzed. Some of the considered plates have been compared with the ANSYS model, and it has been seen that the results are close.

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