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FRACTURE MODE PARTITION IN DELAMINATED BEAMS USING THE CRACK-TIP DISPLACEMENT RATES

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ABSTRACT

The paper presents a method to partition fracture modes in delaminated beams. According to classical laminated beam theory, the axial, shear and bending deformabilities, as well as bending-extension coupling, are taken into account. The kinematics of crack growth is analysed by defining the crack-tip displacement rates as the relative displacements occurring at the crack tip per unit crack extension. Hence, by considering the corresponding work done by the forces exchanged between the separating sub-laminates, explicit expressions for the energy release rate and its mode I and II contributions are deduced.

KEYWORDS

Laminated beams, classical laminated plate theory, delamination, mixed-mode fracture, fracture mode partition, energy release rate, crack-tip displacement rates.

INTRODUCTION

Delamination fracture [1] can be analysed via classical laminated plate theory [2] by modelling delaminated laminates as assemblages of sublaminates, connected by rigid or deformable joints and interfaces. Delamination growth occurs when the energy release rate, G, reaches a critical value, G_c [3]. In general, however, delamination cracks propagate under mixed-mode fracture conditions, so that it is necessary to partition G into two additive contributions, G_I and G_{II}, related to fracture modes I (opening) and II (sliding), respectively [4].

For rigidly connected sublaminates, Williams [5] proposed a *global method* to partition the energy release rate, based on analysis of the global forces acting on the cracked laminate. Schapery and Davidson [6] proposed a method based on classical plate theory. Independently, Suo and Hutchinson developed a *local method* [7], which considers the singular stress field at the crack tip of a semi-infinite crack propagating between two infinite elastic layers. On the other hand, if the sublaminates are connected by a deformable interface, the modal contributions to G can be computed based on the peak values of the interfacial stresses at the crack tip [8–10], or via an adaptation of the local method [11].

This paper presents a method to partition fracture modes in planar laminated beams affected by through-the-width delaminations. To this aim, a delaminated beam is considered as an assemblage of three rigidly connected laminated beams. According to classical laminated beam theory, the axial, shear and bending deformabilities, as well as bending-extension coupling, are taken into account. Under general load conditions, a small extension of the existing crack is considered. The kinematics of crack growth is analysed by defining the *crack-tip displacement rates* as the relative displacements occurring at the crack tip per unit crack extension. Besides, the *crack-tip forces* exchanged between the separating sub-laminates are computed. Lastly, by considering the work done by the crack-tip forces for the corresponding crack-tip displacement rates, explicit expressions for the energy release rate and its modal contributions are deduced.

DELAMINATED BEAM MODELLING

Beam-theory model of a delaminated laminate

Consider a laminate AB of length L, thickness $H = 2\,h$, and width W, affected by a through-the-width delamination of length a (Fig. 1). The delamination runs from the end section A to an intermediate section C to which the crack tip C belongs, thus splitting the laminate into two sublaminates of thicknesses $H_1 = 2\,h_1$ and $H_2 = 2\,h_2$, respectively. The length of the unbroken part of the laminate, included between sections C and B, is $b = L - a$.

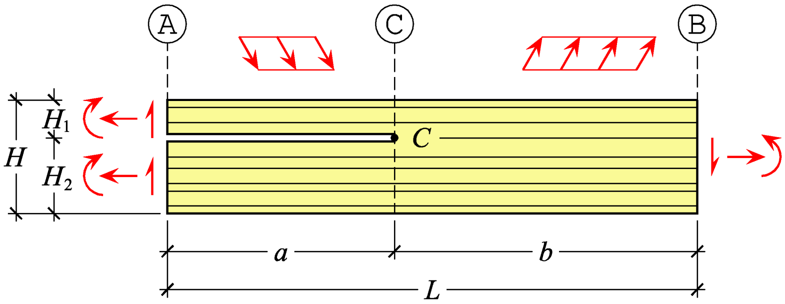

Fig. 1: The delaminated laminate subjected to concentrated and distributed loads

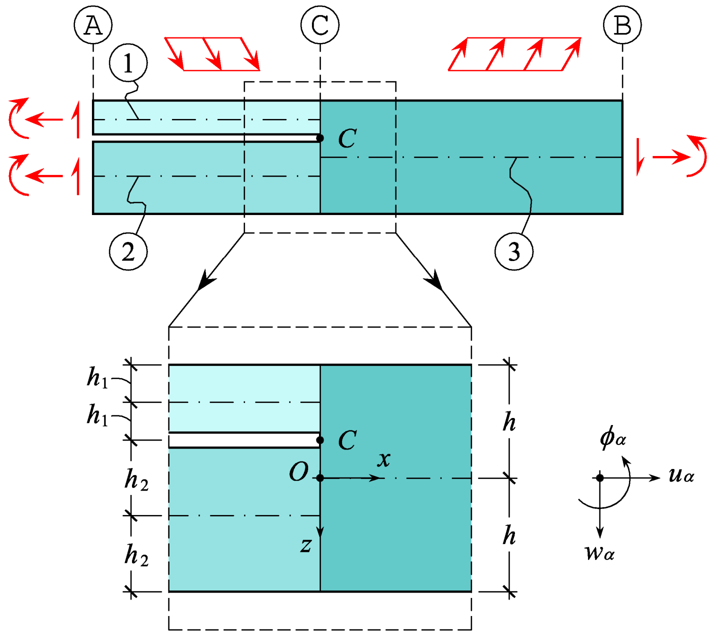

Fig. 2: The delaminated laminate as an assemblage of three laminated beams

The laminate is modelled as an assemblage of three laminated beams (identified by the numbers $\alpha = 1, 2, 3$), each rigidly connected to the others at section C (Fig. 2). Beams No. 1 and 2 coincide with the upper and lower sublaminates, respectively, in the delaminated part of the laminate (between sections A and C). Beam No. 3 coincides with the unbroken part of the laminate (between sections C and B). A rectangular reference system Ozx is fixed with the origin O at the intersection between section C and the centreline of beam No. 3. The x- and z-axes are aligned with the laminate's axial and transverse directions, respectively. Correspondingly, u_α and w_α denote the axial and transverse displacements of the beams' centrelines, and ϕ_α denote the cross sections' rotations, positive if counter-clockwise. The laminate is supposed to be in equilibrium under a given system of in-plane loads.

According to Timoshenko's beam theory, the *axial strain*, *shear strain*, and *curvature* in the sublaminates are

$$\varepsilon_\alpha(x) = \frac{du_\alpha}{dx}, \qquad \gamma_\alpha(x) = \frac{dw_\alpha}{dx} + \phi_\alpha(x), \qquad \kappa_\alpha(x) = \frac{d\phi_\alpha}{dx}. \tag{1}$$

Hence, in line with classical laminated beam theory, the *axial force*, *shear force*, and *bending moment* are

$$N_\alpha(x) = W\,[A_\alpha \varepsilon_\alpha(x) + B_\alpha \kappa_\alpha(x)], \quad Q_\alpha(x) = W\,C_\alpha \gamma_\alpha(x), \quad M_\alpha(x) = W\,[B_\alpha \varepsilon_\alpha(x) + D_\alpha \kappa_\alpha(x)], \tag{2}$$

where A_α, B_α, C_α, and D_α are, respectively, the *extension stiffness*, *bending-extension coupling stiffness*, *shear stiffness*, and *bending stiffness* (per unit width) [2]. For what follows, it is convenient to define also the corresponding *extension compliance*, *bending-extension coupling compliance*, *shear compliance*, and *bending compliance*,

$$a_\alpha = \frac{D_\alpha}{A_\alpha D_\alpha - B_\alpha^2}, \qquad b_\alpha = -\frac{B_\alpha}{A_\alpha D_\alpha - B_\alpha^2}, \qquad c_\alpha = \frac{1}{C_\alpha}, \qquad d_\alpha = \frac{A_\alpha}{A_\alpha D_\alpha - B_\alpha^2}. \tag{3}$$

Crack-tip relative displacements and crack-tip displacement rates

Imagine first that a small segment S of the laminate is cut out in the neighbourhood of the crack tip *C* (Fig. 3a). Regardless of the actual load system applied to the laminate, if no concentrated loads are applied at the crack-tip cross section, the segment S will be in equilibrium under the action of the internal forces applied on the cross sections close to the crack tip. Thus, if N_1, Q_1, M_1 and N_2, Q_2, M_2 respectively denote the internal forces in beams No. 1 and 2 for $x \to 0$, the internal forces in beam No. 3 must be

$$N_3 = N_1 + N_2, \qquad Q_3 = Q_1 + Q_2, \qquad M_3 = M_1 + M_2 - N_1 h_2 + N_2 h_1. \tag{4}$$

Next, suppose that the crack propagates in a self-similar way, increasing its length by a small amount, Δa. Hence, the crack-tip segment S transforms into the segment S_0 (Fig. 3b), where the crack tip reaches a new position, identified by point *D*, and the point *C* splits into two points, C_1 and C_2, belonging to beams No. 1 and 2, respectively. Under fixed load conditions, the internal forces in the cross sections close to the crack tip do not change appreciably, so that S_0 can be considered in equilibrium under the same internal forces acting on S.

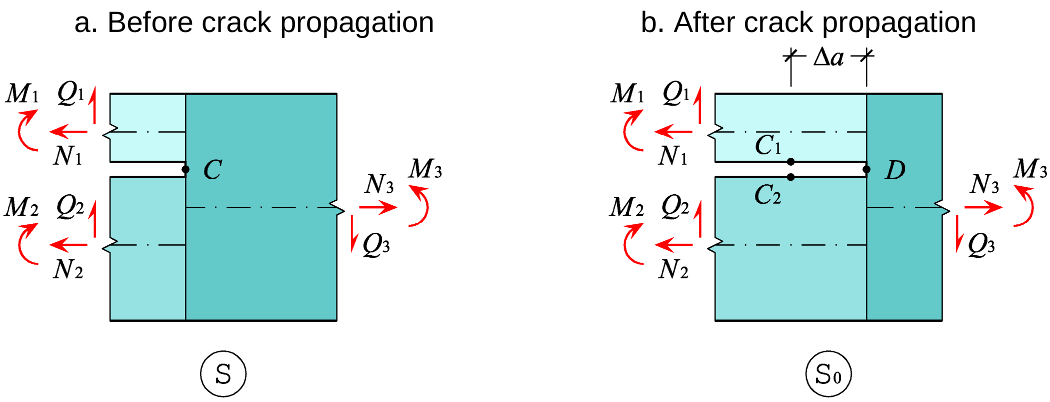

Fig. 3: Elementary segment of the laminate in the neighbourhood of the crack tip

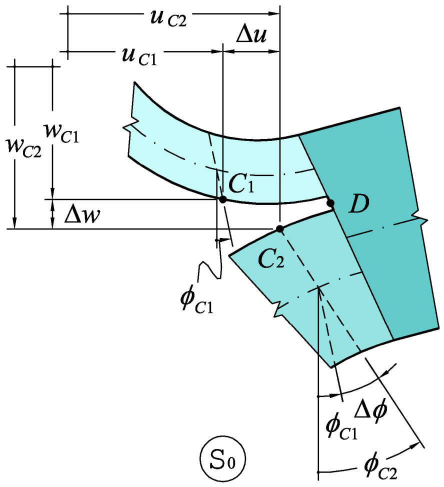

Fig. 4: Crack-tip relative displacements in system S_0

Displacement compatibility, however, is lost when S transforms into S_0, as in the latter system points C_1 and C_2 generally undergo non-zero relative displacements (Fig. 4),

$$\begin{aligned} \Delta u &= u_{C2} - u_{C1} = (u_2 - \phi_2 h_2) - (u_1 + \phi_1 h_1), \\ \Delta w &= w_{C2} - w_{C1} = w_2 - w_1, \\ \Delta\phi &= \phi_{C2} - \phi_{C1} = \phi_2 - \phi_1, \end{aligned} \tag{5}$$

where all the (generalised) displacements are tacitly evaluated at the crack-tip section (x = 0).

By solving an auxiliary problem of a laminated cantilever beam loaded at its end (details are here omitted for brevity), it is easily shown that the *crack-tip relative displacements* are

$$\begin{aligned} \Delta u &\cong \frac{1}{W}[(a_1 + b_1 h_1)\, N_1 - (a_2 - b_2 h_2)\, N_2 + (b_1 + d_1 h_1)\, M_1 - (b_2 - d_2 h_2)\, M_2]\, \Delta a, \\ \Delta w &\cong \frac{1}{W}(c_1 Q_1 - c_2 Q_2)\, \Delta a, \\ \Delta\phi &\cong \frac{1}{W}(b_1 N_1 - b_2 N_2 + d_1 M_1 - d_2 M_2)\, \Delta a, \end{aligned} \tag{6}$$

where higher-order powers of Δa have been neglected. In order to eliminate the dependence on Δa, the *crack-tip displacement rates* are defined as

$$\begin{aligned} \eta_u &= \lim_{\Delta a \to 0} \frac{\Delta u}{\Delta a} = \frac{1}{W}[(a_1 + b_1 h_1)\, N_1 - (a_2 - b_2 h_2)\, N_2 + (b_1 + d_1 h_1)\, M_1 - (b_2 - d_2 h_2)\, M_2], \\ \eta_w &= \lim_{\Delta a \to 0} \frac{\Delta w}{\Delta a} = \frac{1}{W}(c_1 Q_1 - c_2 Q_2), \\ \eta_\phi &= \lim_{\Delta a \to 0} \frac{\Delta \phi}{\Delta a} = \frac{1}{W}(b_1 N_1 - b_2 N_2 + d_1 M_1 - d_2 M_2). \end{aligned} \tag{7}$$

By recalling equations (2) and (3), it can be shown that

$$\eta_u = \varepsilon_1 - \varepsilon_2 + \kappa_1 h_1 + \kappa_2 h_2, \qquad \eta_w = \gamma_1 - \gamma_2, \qquad \eta_\phi = \kappa_1 - \kappa_2, \tag{8}$$

where all the strain measures are evaluated at the crack-tip section (x = 0).

Crack-tip forces

Displacement compatibility of system S can be recovered by superimposing to system S_0 an auxiliary system, S_C, where suitable axial forces, N_C, transverse forces, Q_C, and couples, M_C, are exchanged between points C_1 and C_2 (Fig. 5).

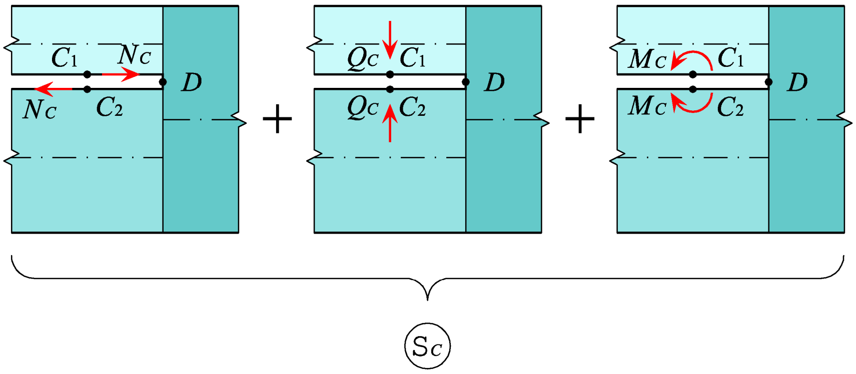

Fig. 5: Crack-tip forces in system S_C.

The intensities of the above *crack-tip forces* are determined in such a way as to restore displacement compatibility previous to crack growth. To this aim, the crack-tip displacement rates associated with system S_C are first computed,

$$\eta_u^C = -N_C\,\eta_u^N - M_C\,\eta_u^M, \qquad \eta_w^C = -Q_C\,\eta_w^Q, \qquad \eta_\phi^C = -N_C\,\eta_\phi^N - M_C\,\eta_\phi^M, \tag{9}$$

where

$$\begin{aligned} &\eta_u^N = (a_1 + a_2 + 2b_1h_1 - 2b_2h_2 + d_1h_1^2 + d_2h_2^2)/W, \quad \eta_u^M = \eta_\phi^N = (b_1 + b_2 + d_1h_1 - d_2h_2)/W, \\ &\eta_\phi^M = (d_1 + d_2)/W, \qquad \eta_w^Q = (c_1 + c_2)/W \end{aligned} \tag{10}$$

are generalised compliances, which describe the deformability of the crack-tip element. Then, compatibility is restored by imposing

$$\eta_u + \eta_u^C = 0, \qquad \eta_w + \eta_w^C = 0, \qquad \eta_\phi + \eta_\phi^C = 0. \tag{11}$$

By substituting equations (9) into (10) and solving, the crack-tip forces are obtained,

$$N_C = \frac{\eta_\phi^M\eta_u - \eta_u^M\eta_\phi}{\eta_u^N\eta_\phi^M - \eta_u^M\eta_\phi^N}, \qquad Q_C = \frac{\eta_w}{\eta_w^Q}, \qquad M_C = \frac{\eta_u^N\eta_\phi - \eta_\phi^N\eta_u}{\eta_u^N\eta_\phi^M - \eta_u^M\eta_\phi^N}. \tag{12}$$

ENERGY RELEASE RATE AND FRACTURE MODE PARTITIONING

Energy release rate

Under fixed displacements [3], the energy release rate associated with crack growth is

$$G = -\frac{1}{W}\lim_{\Delta a \to 0}\frac{\Delta U}{\Delta a}, \tag{13}$$

where ΔU is the change in strain energy related to the increase in crack length Δa. According to the definitions given in the previous section,

$$\Delta U = U_0 - U = -U_C \,, \tag{14}$$

where U, U_0 and U_C are the strain energies in systems S, S_0, and S_C, respectively. In particular, Clapeyron's theorem yields

$$U_C = \frac{1}{2}(N_C\ \Delta u + Q_C\ \Delta w + M_C\ \Delta\phi)\,, \tag{15}$$

where Δu, Δw, and $\Delta\phi$ are the crack-tip relative displacements in system S_0, given by equations (6), which are equal in magnitude and opposite in sign to those caused by the crack-tip forces in S_C. By substituting equations (14) and (15) into (13), and remembering equations (7), the energy release rate is obtained,

$$G = \frac{1}{2W}(N_C\ \eta_u + Q_C\ \eta_w + M_C\ \eta_\phi)\,. \tag{16}$$

Furthermore, by substituting equations (9) and (11) into (16), the energy release rate can be expressed in terms of the crack-tip forces only,

$$G = \frac{1}{2W}[\eta_u^N\ {N_C}^2 + (\eta_\phi^N + \eta_u^M)\, N_C M_C + \eta_\phi^M\ {M_C}^2 + \eta_w^Q\ {Q_C}^2]\,, \tag{17}$$

or, by substituting equations (12) into (17), in terms of the crack-tip displacement rates only,

$$G = \frac{1}{2W}[\frac{\eta_\phi^M {\eta_u}^2 - (\eta_\phi^N + \eta_u^M)\, \eta_u \eta_\phi + \eta_u^N {\eta_\phi}^2}{\eta_u^N \eta_\phi^M - \eta_\phi^N \eta_u^M} + \frac{{\eta_w}^2}{\eta_w^Q}]\,. \tag{18}$$

Fracture mode partitioning

In planar fracture mechanics problems the energy release rate can be decomposed as

$$G = G_{\mathrm{I}} + G_{\mathrm{II}}\,, \tag{19}$$

where the addends G_{I} and G_{II} are related to fracture modes I (opening) and II (sliding), respectively. Clearly, G_{I} should depend on the crack-tip displacement rates η_w and η_ϕ, while G_{II} should depend on η_u. However, some caution is required in order to correctly partition G into its modal contributions. By close examination of equations (17) and (18), it is apparent that the terms depending on Q_C and η_w contribute to G_{I} only (incidentally, these terms are relevant only if shear deformability is considered), while the terms depending on N_C, M_C and η_u, η_ϕ are coupled and, hence, contribute to both G_{I} and G_{II}. To sum up, it is convenient to start from determining the mode II contribution, which must be of the form:

$$G_{\mathrm{II}} = \frac{N_C^{\mathrm{II}}\ \eta_u}{2W}\,, \tag{20}$$

where

$$N_C^{\mathrm{II}} = \eta_u / \eta_u^N \tag{21}$$

is the crack-tip axial force that would cancel the crack-tip sliding displacement rate, η_u, if no crack-tip couple, M_C, were present. Remembering equations (10) and (11), it is apparent that N_C^{II} is in general distinct from N_C, since the latter also contributes to cancel η_ϕ, but they coincide if $\eta_u^M = \eta_\phi^N = 0$. This happens, for instance, when the delaminated sublaminates are uncoupled in bending-extension ($b_1 = b_2 = 0$) and balanced ($d_1h_1 = d_2h_2$). By substituting equation (21) into (20) and (19), the modal contributions to the energy release rate in terms of the crack-tip displacement rates are finally obtained,

$$G_I = \frac{1}{2W}\left[\frac{(\eta_\phi - \frac{\eta_\phi^N}{\eta_u^N}\eta_u)^2}{\eta_\phi^M - \frac{\eta_\phi^N}{\eta_u^N}\eta_u^M} + \frac{\eta_w^{\ 2}}{\eta_w^Q}\right], \qquad G_{II} = \frac{1}{2W}\frac{\eta_u^{\ 2}}{\eta_u^N}. \tag{22}$$

EXAMPLE: THE ADCB TEST

As an example, the presented method is applied to study the asymmetric double cantilever beam (ADCB) test (Fig. 6), usable to measure mixed-mode delamination toughness [10].

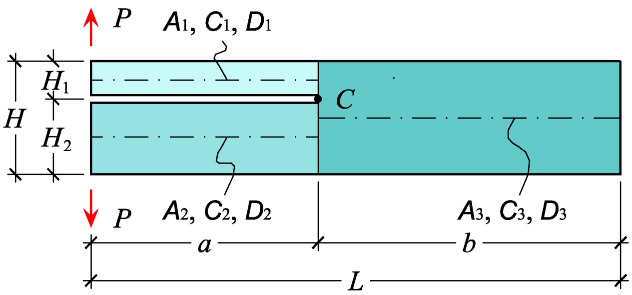

Fig. 6: The asymmetric double cantilever beam (ADCB) test

The internal forces at the crack-tip sections in this case are

$$N_1 = 0, \quad Q_1 = P, \quad M_1 = Pa; \qquad N_2 = 0, \quad Q_2 = -P, \quad M_2 = -Pa. \tag{23}$$

Equations (7) yield the crack-tip displacement rates,

$$\eta_u = Pa\,(d_1h_1 - d_2h_2)/W, \qquad \eta_w = P\,(c_1 + c_2)/W, \qquad \eta_\phi = Pa\,(d_1 + d_2)/W, \tag{24}$$

and equations (10) furnish the crack-tip compliances,

$$\eta_u^N = (a_1 + a_2 + d_1h_1^2 + d_2h_2^2)/W, \quad \eta_u^M = \eta_\phi^N = (d_1h_1 - d_2h_2)/W, \quad \eta_\phi^M = (d_1 + d_2)/W, \quad \eta_w^Q = (c_1 + c_2)/W. \tag{25}$$

Lastly, equations (24) yield the mode I and II contributions to the energy release rate,

$$G_I = \frac{P^2a^2}{2W^2}\left[d_1 + d_2 - \frac{(d_1h_1 - d_2h_2)^2}{a_1 + a_2 + d_1h_1^2 + d_2h_2^2}\right] + \frac{P^2}{2W^2}(c_1 + c_2), \quad G_{II} = \frac{P^2a^2}{2W^2}\frac{(d_1h_1 - d_2h_2)^2}{a_1 + a_2 + d_1h_1^2 + d_2h_2^2}. \tag{26}$$

It is interesting to notice that equations (26) are identical with those that can be obtained from the elastic-interface model of the ADCB test [10] in the limit case of a rigid interface.

CONCLUSIONS

A novel method to partition fracture modes in planar laminated beams affected by through-the-width delaminations has been presented. According to classical laminated beam theory, the axial, shear and bending deformabilities are taken into account. Moreover, the method considers *bending-extension coupling*, which apparently had not been contemplated before by the fracture mode partition methods proposed in literature. The kinematics of crack growth is analysed by introducing the *crack-tip displacement rates*, defined as the relative displacements occurring at the crack tip per unit crack extension. These quantities have been used to determine explicit expressions for the crack-tip forces, energy release rate and its modal contributions in delaminated beams.

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