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Evolutionary equilibrium paths of statically and kinematically indeterminate reticulated deployable structures

S.S. Ligarò, P.S. Valvo
University of Pisa
Via Diotisalvi 2, I 56126 Pisa (PI), Italy
s.ligaro@ing.unipi.it

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Abstract

In this paper, we present a numerical algorithm for tracing the equilibrium paths of simultaneously statically and kinematically indeterminate structures. The method is suitable for analysing reticulated deployable structures, and enables monitoring their evolution during the setting up. The main features of the method are illustrated through a simple example relative to a crank gear model.

1 Introduction

Deployable structures are a fascinating class of mechanical systems. In their original configuration they are unable to sustain applied loads, due to their kinematical indeterminacy. They become statically efficient just at a quite different final configuration where they acquire a relevant stiffness and load bearing capacity. A similar behaviour is shared with tensegretic structures, inflatable membranes, and many other innovative structures, whose use is intensively growing in modern buildings and civil constructions.

These structures propose unusual structural problems, whose solution demands new analysis tools and non-conventional solution methods. In particular, static and kinematic indeterminacy, usually treated separately, must be taken into account simultaneously. The earliest studies on this argument are due to Kuznetsov [1975], Tarnai [1980] and Pellegrino & Calladine [1986], while more recent contributions are quoted in the paper by Kumar and Pellegrino [2000].

The above systems come out from an 'assemblage' of some elementary components, which in the original configuration is characterised by one or more degrees of freedom. In the deployed configuration these are lost and the structure as a whole is unloaded, while the composing elements usually are in a state of pre-stress [Smaili & Motro 2005]. The equilibrium configurations assumed by a mechanical system can be plotted in the space of the generalised loads and displacements as a set of curves called the 'equilibrium path' [Crisfield 1991]. The paths of deployable structures are characterised by kinematic branches, which represent their deployment mechanism.

The Authors [1999, 2002] have proposed a path-tracing method which has proved to be effective and computationally efficient also in severe circumstances. In this paper, the method is modified in order to analyse the equilibrium paths of reticulated deployable structures. Particular attention is devoted to the detection of critical points and to the determination of kinematic branches.

The main features of the algorithm are illustrated through the study of a simple crank gear model.

2 The equilibrium path

The configurations assumed by a structure subjected to proportional loading are described by a vector of *nodal displacements*, $\mathbf{q}$, which is solution of the non-linear equilibrium equation set

$$\mathbf{f}(\lambda;\mathbf{q}) = \mathbf{D}(\mathbf{q})\,\mathbf{q} - \lambda\mathbf{p} = \mathbf{0}, \tag{1}$$

where $\mathbf{D}(\mathbf{q})$ is the *secant stiffness matrix* of the structure, λ is the *load multiplier* and $\mathbf{p}$ is the *reference load vector*. The solutions of equations (1) can be plotted as a set of curves in the (n+1)-dimensional space spanned by λ and by the components of $\mathbf{q}$, called the *equilibrium path* of the structure. By convention, the curve passing through the *origin*, $\mathbf{t}_0 = [\ 0;\ \mathbf{0}\ ]$, is called *primary branch*, while curves intersecting it, if any, are called *secondary branches*.
In the present work, equations (1) are solved by means of a predictor-corrector scheme of the 'arc-length' family, based on the Newton-Raphson method [Ligarò & Valvo 1999]. The equilibrium path is obtained as a broken line of chords whose endpoints correspond to increasing values of the curvilinear abscissa, s. At a point, $\mathbf{t}(s) = [\lambda(s);\mathbf{q}(s)]$, the *unit tangent vector* to the path, $\dot{\mathbf{t}} = [\dot{\lambda};\dot{\mathbf{q}}]$, is determined by solving the equation set

$$\begin{cases} \mathbf{K}(\mathbf{q})\,\dot{\mathbf{q}} - \dot{\lambda}\mathbf{p} = \mathbf{0}, \\ \dot{\lambda}^2 + \dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}} = 1, \end{cases} \tag{2}$$

where $\mathbf{K}(\mathbf{q}) = \partial[\mathbf{D}(\mathbf{q})\mathbf{q}]/\partial\mathbf{q}$ is the *tangent stiffness matrix* of the structure.
Equations (2) are solved by diagonalising the tangent stiffness matrix through the Jacobi algorithm. In fact, since $\mathbf{K}$ is a symmetric and real-valued matrix, n mutually orthogonal *eigenvectors* exist, $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$, such that

$$\mathbf{K}\mathbf{a}_i = \omega_i\mathbf{a}_i, \quad i = 1, 2, \ldots, n, \tag{3}$$

relative to n real *eigenvalues*, ω_1, ω_2, …, ω_n.
By expressing $\dot{\mathbf{q}}$ with respect to the eigenvector basis,

$$\dot{\mathbf{q}} = \sum_{i=1}^{n} \dot{u}_i \mathbf{a}_i\,, \tag{4}$$

where $\dot{u}_i = \dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{a}_i$, $i = 1, 2, \ldots n$, system (2) can be put in the following form

$$\begin{cases} \omega_i\,\dot{u}_i - \dot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_i = 0, \quad i = 1, 2, \ldots, n, \\ \dot{\lambda}^2 + \sum_{i=1}^{n} \dot{u}_i^{\,2} = 1. \end{cases} \tag{5}$$

3 Point classification

At a *regular point*, all eigenvalues are non-zero. Thus, the unit tangent vector can be determined by solving first system (5) as follows

$$\begin{cases} \dot{\lambda} = \pm[1 + \sum_{i=1}^{n} (\mathbf{p}^{\mathrm{T}}\mathbf{a}_i/\omega_i)^2]^{-1/2}, \\ \dot{u}_i = \dot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_i/\omega_i, \quad i = 1, 2, \ldots, n, \end{cases} \tag{6}$$

and then making use of equations (4) to deduce $\dot{\mathbf{q}}$ (the sign denotes the direction along the path).
At a simple *critical point*, one eigenvalue is zero. Without loss of generality, we suppose that $\omega_1 = 0$, while $\omega_i \neq 0$ for $i > 1$. Solution of system (5) requires three cases to be considered:
a) if $\mathbf{p}^{\mathrm{T}}\,\mathbf{a}_1 \neq 0$ then the critical point is a *limit point*, the tangent to the path is unique and is given by

$$\begin{cases} \dot{\lambda} = 0, \\ \dot{u}_1 = \pm 1, \\ \dot{u}_i = 0, \quad i = 2, \ldots, n; \end{cases} \tag{7}$$

b) if $\mathbf{p}^{\mathrm{T}}\,\mathbf{a}_1 = 0$ and $\dot{\omega}_1 \neq 0$ then the critical point is a *bifurcation point*, and two distinct tangents to the path are present

$$\begin{cases} \dot{\lambda} = \pm[1 + (\mathbf{p}^{\mathrm{T}}\mathbf{A}\,\dot{\mathbf{a}}_1/\dot{\omega}_1)^2 + \sum_{i=2}^{n}(\mathbf{p}^{\mathrm{T}}\mathbf{a}_i/\omega_i)^2]^{-1/2}, \\ \dot{u}_1 = \dot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{A}\,\dot{\mathbf{a}}_1/\dot{\omega}_1\,, \\ \dot{u}_i = \dot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_i/\omega_i, \quad i = 2,\ldots,n, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\lambda} = 0, \\ \dot{u}_1 = \pm 1, \\ \dot{u}_i = 0, \quad i = 2,\ldots,n, \end{cases} \tag{8a,b}$$

where $\mathbf{A} = \sum_{i=1}^{n}\mathbf{a}_i\mathbf{a}_i^{\mathrm{T}}$, while $\dot{\omega}_1$ and $\dot{\mathbf{a}}_1$ are the derivatives of the zero eigenvalue and of its related eigenvector with respect to the curvilinear abscissa, s;

c) if $\mathbf{p}^{\mathrm{T}}\,\mathbf{a}_1 = 0$ and $\dot{\omega}_1 = 0$ then the critical point actually is a *regular point of a kinematic branch*, and the tangent to the path is again given by eqns. (7).

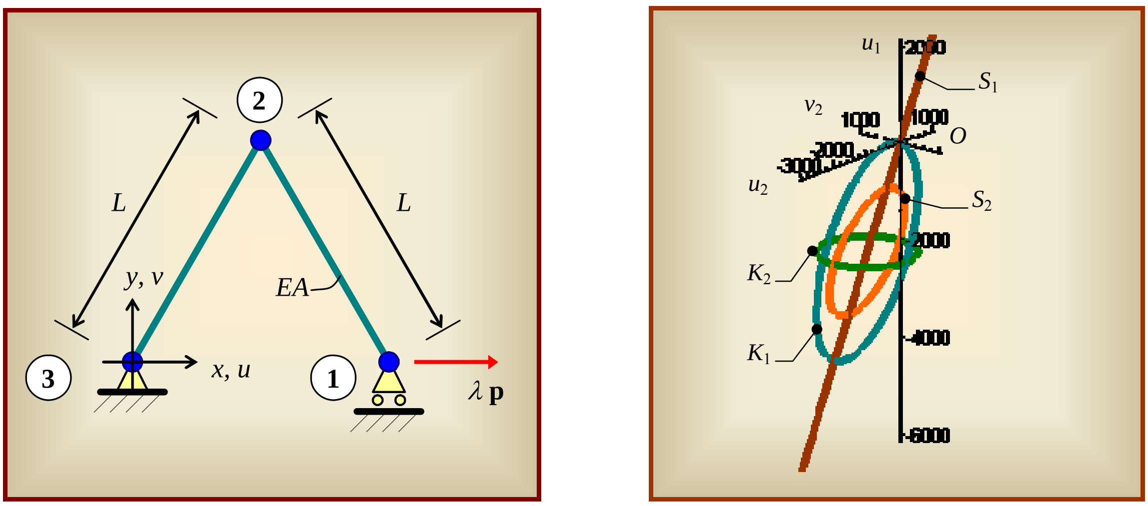
At a double *critical point*, two eigenvalues are zero, say $\omega_1 = 0$ and $\omega_2 = 0$, while $\omega_i \neq 0$ for $i > 2$. Among all the possibilities, here we restrict our attention to the case of a *hill-top branching point*, namely a compound critical point which is simultaneously a limit and a bifurcation point. In this case, two tangents to the path are present, and their expressions are

$$\begin{cases} \dot{\lambda} = 0, \\ \dot{u}_1 = \ddot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_1/\dot{\omega}_1\,, \\ \dot{u}_2 = \ddot{\lambda}\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_2/\dot{\omega}_2\,, \\ \dot{u}_i = 0, \quad i = 3,\ldots,n, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\lambda} = 0, \\ \dot{u}_1 = \sin\alpha, \\ \dot{u}_2 = \cos\alpha, \\ \dot{u}_i = 0, \quad i = 3,\ldots,n, \end{cases} \tag{9a,b}$$

where $\ddot{\lambda} = \pm|\dot{\omega}_1\,\dot{\omega}_2|\,[(\dot{\omega}_1\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_2)^2 + (\dot{\omega}_2\,\mathbf{p}^{\mathrm{T}}\mathbf{a}_1)^2]^{-1/2}$ and $\alpha = \arctan(-\mathbf{p}^{\mathrm{T}}\mathbf{a}_2/\mathbf{p}^{\mathrm{T}}\mathbf{a}_1)$.

4 Application

As a representative example of a deployable structure, we examine the mechanical behaviour of the simple crank gear depicted in Figure 1a. In the original (undeployed) configuration the two bars are superimposed, while in the final (deployed) configuration the bars are collinear. The load, $\lambda\mathbf{p}$, may increase only starting from the latter situation, which will be assumed as the *reference configuration*.



a) b)

Figure 1. Crank gear model: a) structural scheme (L = 100 cm, EA = 105 kN, p = –100 kN); b) the equilibrium path in the u_1-u_2-v_2 space.

Despite the apparent simplicity of the system, its equilibrium path (Figure 1b) exhibits a high degree of complexity. The path is composed of four branches: two static branches, S_1 and S_2, and two kinematic ones, K_1 and K_2. By convention, S_1 is the primary branch because it passes through the origin, O. K_1 and S_2 are secondary branches intersecting the primary one. K_2 can be classified as a *tertiary branch*, since it intersects both the secondary branches but not the primary one.

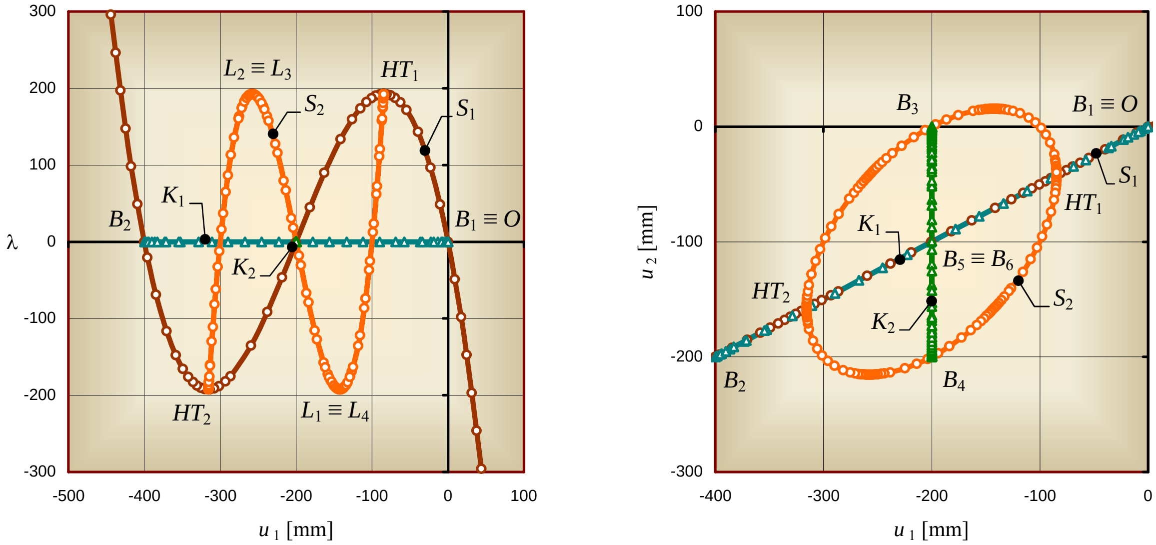


Figure 2. Plane views of the equilibrium path: a) on the u_1-λ plane; b) on the u_1-u_2 plane.

Figure 2a and 2b show two plane views of the equilibrium path as furnished by our algorithm. All the considered types of critical points are present. In particular, the primary branch S_1 intersects the kinematic branch K_1 at the bifurcation points $B_1 \equiv O$ and B_2, and the static branch S_2 at the hill-top points HT_1 and HT_2. The secondary branch S_2 possesses four limit points denoted by L_1, L_2, L_3, and L_4. The tertiary branch K_2 intersects the static branch S_2 at the bifurcation points B_3 and B_4, and the kinematic branch K_1 at the bifurcation points B_5 and B_6.

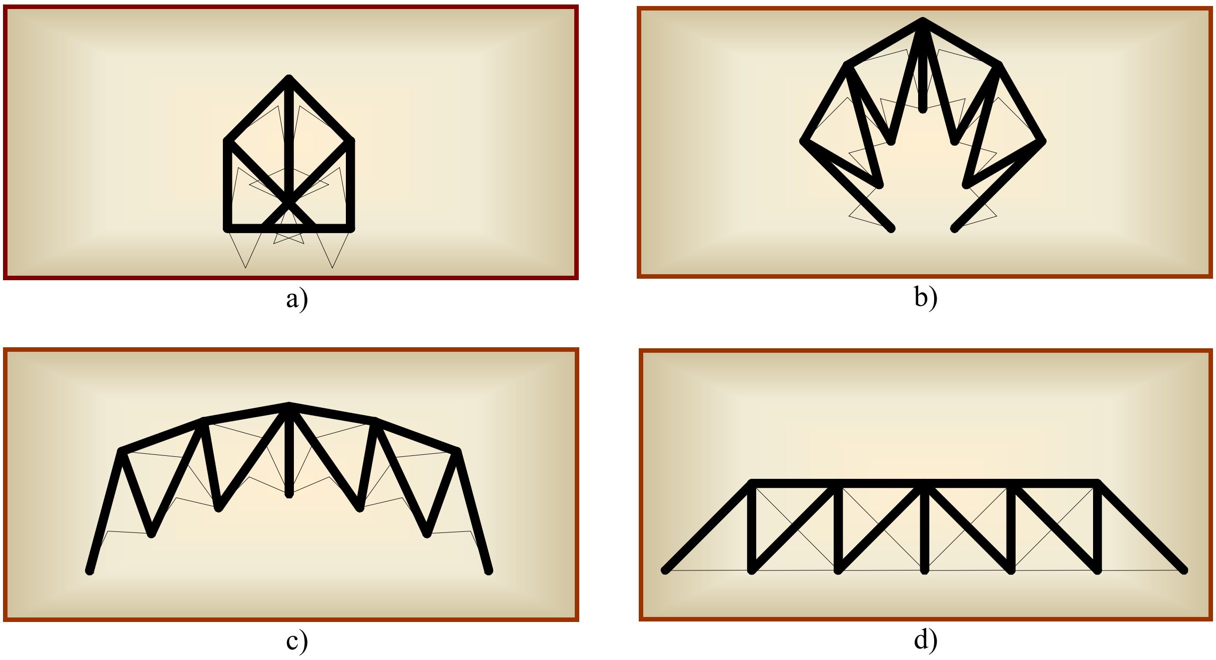


Figure 3. A deployable truss: a) original configuration; b)-c) intermediate configurations; d) final (deployed) configuration.

The points of the kinematic branches are all characterised by the conditions $\lambda = \dot{\lambda} = 0$ and $\omega_1 = \dot{\omega}_1 = 0$. Each of them represents a finite mechanism of the structure. In particular, K_1 corresponds to the proper crank gear mechanism, where joint-1 moves freely back and forth along the *x*-axis, while joint-2 describes a circle of radius *L*. K_2 corresponds to a mechanism where joint-1 is fixed together with joint-3 at the origin, while joint-2 again describes a circle of radius *L*.
The many different mechanical responses of this example may put in serious difficulty most commercial codes for structural analysis, especially when the detection and description of the kinematic branches are required. Therefore, we believe that the above model may represent a valid benchmark test for any algorithm of non-linear structural analysis.

The considered model can also be seen as an elementary component of more complex deployable structures, such as the truss schematised in Figure 3. The figure represents some of the shapes assumed by the system during its deployment. In particular, Figure 3a shows the original (undeployed) configuration, Figures 3b and 3c are relative to two intermediate states, and Figure 3d shows the final (deployed) configuration. Apart from the complexity stemming out from the increased number of degrees of freedom, the study of such structures does not differ conceptually from the simpler previously examined case.

5 Conclusions

In this paper, a numerical algorithm for tracing the equilibrium paths of simultaneously statically and kinematically indeterminate structures was presented. The algorithm is capable of determining the response of reticulated deployable structures, and of monitoring their evolution during the setting up.
The effectiveness of the method was tested through the analysis of a simple crank gear model, whose equilibrium path features both static and kinematic branches, variously intersecting each other, and a wide gamut of critical points. More complex cases of reticulated deployable structures are under consideration.

6 References

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