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NON-LINEAR ANALYSIS OF SOFT ELASTIC MEMBRANES CONTAINING HOLES AND SLITS

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SOMMARIO
Si valuta lo stato di tensione intorno a discontinuità geometriche, quali fori e fessure, in membrane elastiche di materiale soffice soggette a trazione monoassiale. La ricerca delle soluzioni considera in modo sufficientemente accurato le principali fonti di non linearità simultaneamente presenti. In particolare, grandi spostamenti e grandi deformazioni sono tenuti in conto attraverso la misura di deformazione di Green-Lagrange, mentre la non linearità fisica è introdotta dalla legge costitutiva di Ogden. In questo contesto, gli effetti di possibili fenomeni d'instabilità locale sono tenuti in conto automaticamente ricorrendo al concetto di *energia rilassata* proposto da Pipkin per lo studio delle membrane corrugate. Infine, una procedura di continuazione di tipo *arc-length* permette il tracciamento del percorso d'equilibrio. Il modello risultante può essere utile per analizzare la risposta istantanea di taluni tessuti biologici sotto sforzo crescente. Le applicazioni numeriche riguardano casi di membrane rettangolari.

ABSTRACT
We study the stress distribution in the neighbourhood of geometric discontinuities, such as holes and slits, in soft elastic membranes subject to increasing uniaxial traction. The search for the solutions is carried out taking simultaneously into account the main sources of non-linearity. In particular, large displacements and large deformations are considered by adopting the Green-Lagrange measure of strain, while physical non-linearity is entered via Ogden's constitutive law. Within this context, possible phenomena of local buckling are automatically taken into account through the concept of *relaxed energy* proposed by Pipkin for the analysis of wrinkled membranes. Finally, a continuation procedure of the *arc-length* type permits the tracing of the equilibrium path. The proposed model can be usefully exploited for analysing the instantaneous response of biological tissues under increasing loads. Cases of rectangular membranes are illustrated as a numerical application.

1. INTRODUCTION

Membranes made of soft materials, such as thin rubber sheets or biological tissues, usually undergo large displacements and large strains, even under modest tensile stresses. On the other hand, if one attempts to reverse the sign of the in-plane applied loads, he will notice how these elements prefer undergo large out-of-plane displacements (buckling) rather than accepting the introduction of any appreciable superficial contractions. In these cases, compressive stresses cannot exceed a very small value. Rather, an acceptable simplifying assumption is that these exactly vanish throughout the membrane. This hypothesis is here considered to hold.

Because of this peculiar behaviour, shear states of stress would cause a full or partial wrinkling of the membrane. Likewise, geometrical discontinuities, such as holes or slits, determine a local buckling of their surrounding regions even under prevailing tensile state of stress (Cherepanov, [1]). However, since the resulting stress distribution is everywhere quite different from that predicted by standard Membrane Theory, we can reasonably argue that, in soft materials, the stress-concentration around geometrical discontinuities may result less severe.

In order to verify the above statement, a number of different phenomena must be taken simultaneously into account since several sources of non-linearity here coexist. The mechanical model we are going to present deals with each of them through a specific tool. In particular, large displacements and large deformations are considered by adopting the Green-Lagrange measure of strain. Physical non-linearity is taken into account by assuming the membrane to be made of an isotropic hyper-elastic material whose energy density function is of the type proposed by Ogden [2]. Buckling and eventual wrinkling are entered as a physical non-linearity modifying the energy density function, according to the *relaxed energy* concept proposed by Pipkin [3]. Finally, evolution of the whole phenomenon is analysed via a path-tracing procedure of the arc-length type [5].

The set of equations that govern the non-linear equilibrium of initially flat elastic membranes under in-plane loads is derived via the principle of stationary total potential energy. Solutions are carried out numerically in a finite element analysis context, based on a total Lagrangian formulation. An *ad hoc* isoparametric triangular element was implemented, able to account for the above sources of non-linearity.

Numerical results relating to rectangular membranes appear to confirm our initial assertion.

2. PROBLEM FORMULATION

2.1. Geometry

Consider an elastic membrane, which in the reference configuration, $\mathcal{C}^*$, occupies the region Ω of the plane *OXY* bounded by the curve $\mathbf{\Gamma}$. Let $\mathbf{\Gamma}_u$ be the part of $\mathbf{\Gamma}$ where displacements, $\mathbf{u} = \bar{\mathbf{u}}$, are prescribed and $\mathbf{\Gamma}_p$ the part where an in-plane edge traction, $\mathbf{t} = \mu\, \bar{\mathbf{t}}$, is assigned. We suppose that loads increase proportionally to the single multiplier, $\mu \in \Re^+$. Finally, let the membrane be endowed with a central hole or a slit, whose boundary, $\mathbf{\Gamma}_i$, is traction free.

2.2. Reduction to a planar problem

Let $\mathbf{X} = [X, Y, 0]^{\mathrm{T}}$ be the position vector of a material point, $P \in \Omega$, of the membrane in the reference configuration, $\mathcal{C}^*$. In the generic configuration, $\mathcal{C}$, the point occupies the new position, $\mathbf{x} = [x, y, z]^{\mathrm{T}}$, thus undergoing the total displacement, $\mathbf{U} = \mathbf{x} - \mathbf{X} = [u, v, w]^{\mathrm{T}}$.

In presence of buckling or wrinkling, the membrane reaches its equilibrium in a deformed configuration, which is quite far away from the initial one. Therefore, a large displacement analysis is usually required. Furthermore, if the material is soft enough, then we must take into account the contemporary presence of large strains. For these reasons, the non-linear equilibrium problem should be based on a suitable measure of strain. Here, we consider the Green-Lagrange tensor

$$\mathbf{E} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right) + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)\right], \tag{1}$$

where $\partial \mathbf{U} / \partial \mathbf{X}$ denotes the displacement gradient.

At equilibrium, because of its limited thickness, the membrane will be in a prevailing generalised plane state of stress. Therefore, the components of strain to be determined reduce to E_{XX}, E_{YY}, and $G_{XY} = 2E_{XY}$ since E_{ZZ} is a function of the preceding ones and $E_{XZ} = E_{YZ} = 0$.

The total strain vector, $\mathbf{e} = [E_{XX}, E_{YY}, G_{XY}]^{\mathrm{T}}$, can be written out as

$$\mathbf{e} = \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^w, \tag{2}$$

where $\boldsymbol{\varepsilon}$ represents the contribution of the in-plane displacements, *u* and *v*, while the contribution of the out-of-plane displacement, *w*, is expressed by the vector $\boldsymbol{\varepsilon}^w$ whose components are the so-called *wrinkle strains* [4]. However, by means of the concept of *relaxed energy*, as given by Pipkin, local buckling and wrinkling phenomena can be taken automatically into account as a physical non-linearity. For this reason, the vector $\boldsymbol{\varepsilon}^w$ will be omitted in the following analysis, so that the problem is reduced to a planar one.

With reference to the components of $\boldsymbol{\varepsilon}$, the in-plane principal strains, E_1 and E_2, result

$$E_{1,2} = \frac{1}{2}\left(E_{XX} + E_{YY} \pm \sqrt{(E_{XX} - E_{YY})^2 + G_{XY}{}^2}\right), \tag{3}$$

while the in-plane principal stretches, λ_1 and λ_2, are given by

$$\lambda_1{}^2 = 2E_1 + 1, \qquad \lambda_2{}^2 = 2E_2 + 1. \tag{4}$$

2.3. Material non-linearity

If the material of the membrane is capable of experiencing finite deformations, the region surrounding a geometrical discontinuity can more easily adapt its shape to the state of stress there present. In many cases, this phenomenon may lead to a relevant reduction of the severity of stress-concentration. Here, we suppose the membrane to be made of a homogeneous isotropic hyper-elastic material whose energy density function is of the type proposed by Ogden. This material appears appropriate for our purposes, because of its good agreement with experimental data on rubber over a wide range of tensile strain.

The strain energy density, as a function of the principal stretches, has the form

$$\hat{\omega}(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1{}^{\alpha_p} + \lambda_2{}^{\alpha_p} + \lambda_3{}^{\alpha_p} - 3\right). \tag{5}$$

Limiting the above summation to $N = 3$ and adopting the hypothesis of material incompressibility

$$\lambda_1 \lambda_2 \lambda_3 = 1, \tag{6}$$

one obtains the expression

$$\omega(\lambda_1, \lambda_2) = \sum_{p=1}^{3} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + (\lambda_1 \lambda_2)^{-\alpha_p} - 3 \right). \tag{7}$$

2.4. Admissible values of the in-plane principal stretches

Based on the definition of *natural width* (see Pipkin, [3]), a qualitative prevision of the type of equilibrium state existing in the neighbourhood of a point *P* of the membrane becomes possible. To this aim, the in-plane principal stretches, λ_1 and λ_2, play the role of reference variables. For an incompressible material, we will say that the point *P* belongs to a *taut* region if there λ_1 and λ_2 satisfy the conditions

$$\lambda_1 \geq \lambda_2^{-1/2} \text{ and } \lambda_2 \geq \lambda_1^{-1/2}. \tag{8}$$

Vice versa, the point *P* belongs to a *wrinkled* region if there

$$\lambda_1 > 1 \text{ and } 0 < \lambda_2 < \lambda_1^{-1/2}, \text{ or, alternatively, if } \lambda_2 > 1 \text{ and } 0 < \lambda_1 < \lambda_2^{-1/2}, \tag{9}$$

Finally, we will consider the point *P* belonging to an *inactive* (*slack* or *buckled*) region if there

$$0 < \lambda_1 \leq 1 \text{ and } 0 < \lambda_2 \leq 1. \tag{10}$$

2.5. The concept of *relaxed energy*

By the hypothesis of material incompressibility, the expressions of the principal Biot stresses, t_1 and t_2, which are work-conjugate with the principal stretches, λ_1 and λ_2, are

$$\begin{aligned} t_1 &= \frac{\partial \omega}{\partial \lambda_1} = \frac{1}{\lambda_1} \sum_{p=1}^{3} \mu_p \left[\lambda_1^{\alpha_p} - (\lambda_1 \lambda_2)^{-\alpha_p} \right], \\ t_2 &= \frac{\partial \omega}{\partial \lambda_2} = \frac{1}{\lambda_2} \sum_{p=1}^{3} \mu_p \left[\lambda_2^{\alpha_p} - (\lambda_1 \lambda_2)^{-\alpha_p} \right]. \end{aligned} \tag{11}$$

Depending on the values assumed by λ_1 and λ_2, these stresses may be positive or negative, in the latest case contradicting the stated inability of the membrane of sustaining any compressive stress. The simplest manner to obviate to this drawback consists in using for ω in (11) the following *relaxed energy*, ω_{rel}, defined as

$$\omega_{rel}(\lambda_1,\lambda_2)=\begin{cases} 0, & 0<\lambda_1<1 \text{ and } 0<\lambda_2<1,\\ \omega(\lambda_1,\lambda_1^{-1/2}), & \lambda_1\geq 1 \text{ and } 0<\lambda_2<\lambda_1^{-1/2},\\ \omega(\lambda_2^{-1/2},\lambda_2), & 0<\lambda_1<\lambda_2^{-1/2} \text{ and } \lambda_2\geq 1,\\ \omega(\lambda_1,\lambda_2), & \lambda_1\geq\lambda_2^{-1/2} \text{ and } \lambda_2\geq\lambda_1^{-1/2}, \end{cases} \tag{12}$$

which automatically takes into account the effects on stress due to local buckling and wrinkling, setting to zero any compressive stress arising in wrinkled and inactive regions. In other words, use of relation (12), permits local buckling and wrinkling phenomena to be formally treated as a physical non-linearity.

3. THE STATIONARY POTENTIAL ENERGY PRINCIPLE

By using the previous expressions for the relaxed energy density, ω_{rel}, we can define the total potential energy of the system (membrane plus loads) in the actual configuration, $\mathcal{C}$, as

$$\Pi(u,v,\mu)=\int_{V_0}\omega_{rel}dV-\mu\int_{\Gamma_p}\bar{\mathbf{t}}\cdot\mathbf{u}\,d\Gamma \tag{13}$$

where $\mathbf{u}=[u,v]^{\mathrm{T}}$ denotes the in-plane displacement vector, while V_0 and $\boldsymbol{\Gamma}_p$ refer to the volume and the loaded boundary of the membrane in the reference configuration, $\mathcal{C}^*$, respectively.

For a given load parameter, μ, equilibrium of the system corresponds to stationary points for the functional (13). Then, solving the system of equilibrium equations for increasing values of the load multiplier, μ, permits the tracing of the equilibrium path, which fully describes the evolution of the phenomenon. However, except for a few elementary cases, the stated problem can be handled in practice only by recourse to numerical procedures. Among these, we believe that incremental-iterative strategies, such as *arc-length* methods, should be privileged since they permit a step-wise accurate control of the equilibrium conditions [5].

In addition to what presented up to now, using an arc-length method requires further knowledge of the so-called tangent stiffness matrix of the discretised model. Details of the adopted numerical strategy are here omitted for brevity and will be fully given in an incoming paper. Here we limit ourselves to the presentation of some results obtained till now.

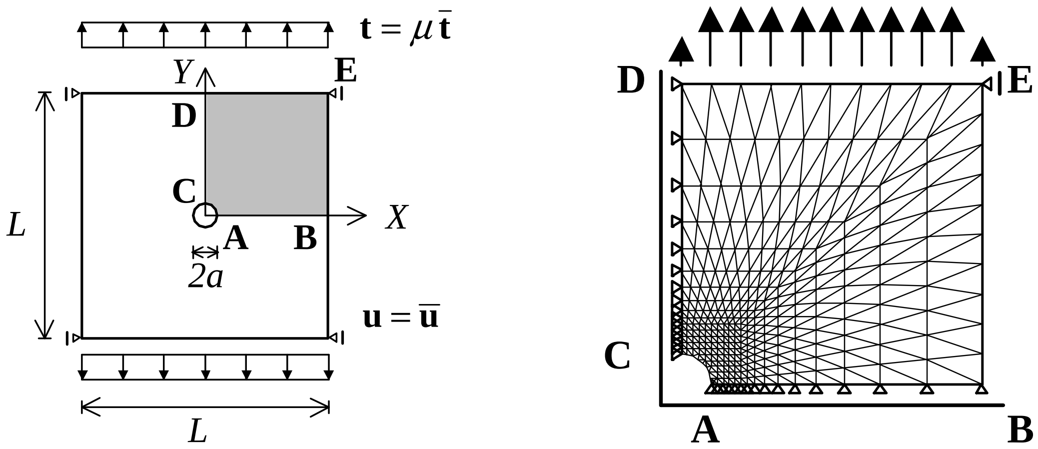
4. A SQUARE MEMBRANE WITH A CENTRAL CIRCULAR HOLE

The model is first applied to the analysis of an initially flat square membrane endowed with a central circular hole (Figure 1a), under prevailing uniaxial state of stress. The edge length is $L=100\,mm$, the thickness is $h=1\,mm$, and the hole has diameter $2a=10\,mm$. An increasing uniform traction $\mathbf{t}=\mu\bar{\mathbf{t}}=\mu[0,S_\infty]^{\mathrm{T}}$, where $S_\infty=0.2\,N$, is applied along the edges parallel to the *X*-axis, while the other edges, $X=\pm L/2$, are traction free. Finally, the *u*-displacement of the corner points is inhibited.

On accounting for the existing symmetries, only a quarter of the membrane needs being considered. The resulting FEM model is represented in Figure 1b. It comprises 316 nodes and 568 constant strain/constant stress triangular elements. Moreover, the following values

$$\mu_1 = 0.63\,MPa\,, \quad \mu_2 = 0.0012\,MPa\,, \quad \mu_3 = -0.01\,MPa\,,$$
$$\alpha_1 = 1.3\,, \quad \alpha_2 = 5.0\,, \quad \alpha_3 = -2.0\,, \tag{14}$$

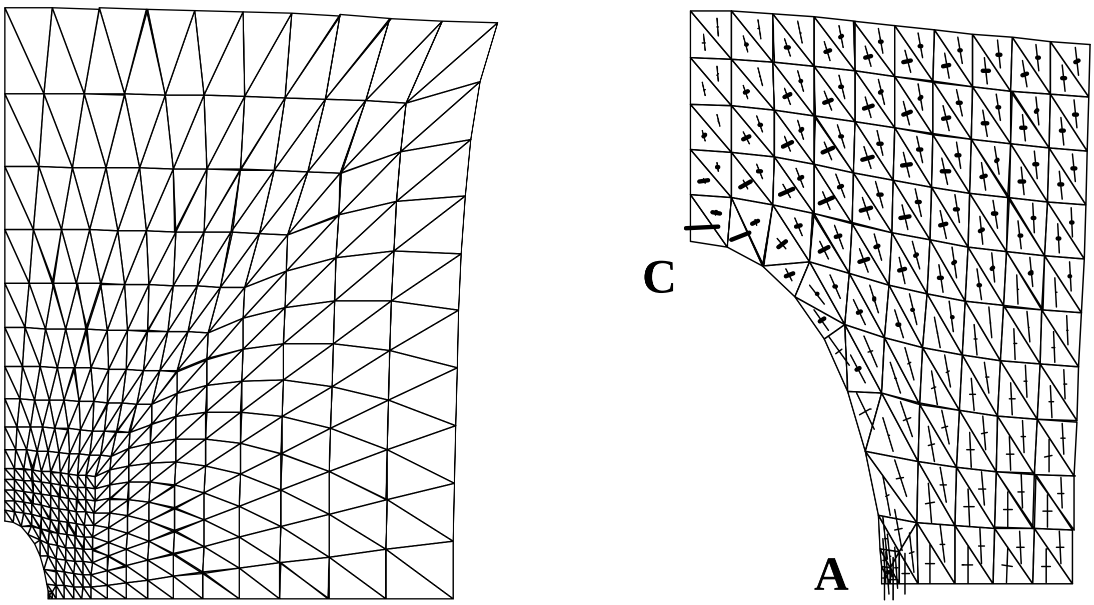
were used for the parameters appearing in the expression (7) of the strain-energy.



a) b)

Figure 1 – A square membrane with a central circular hole: a) geometry and loads; b) FEM model.

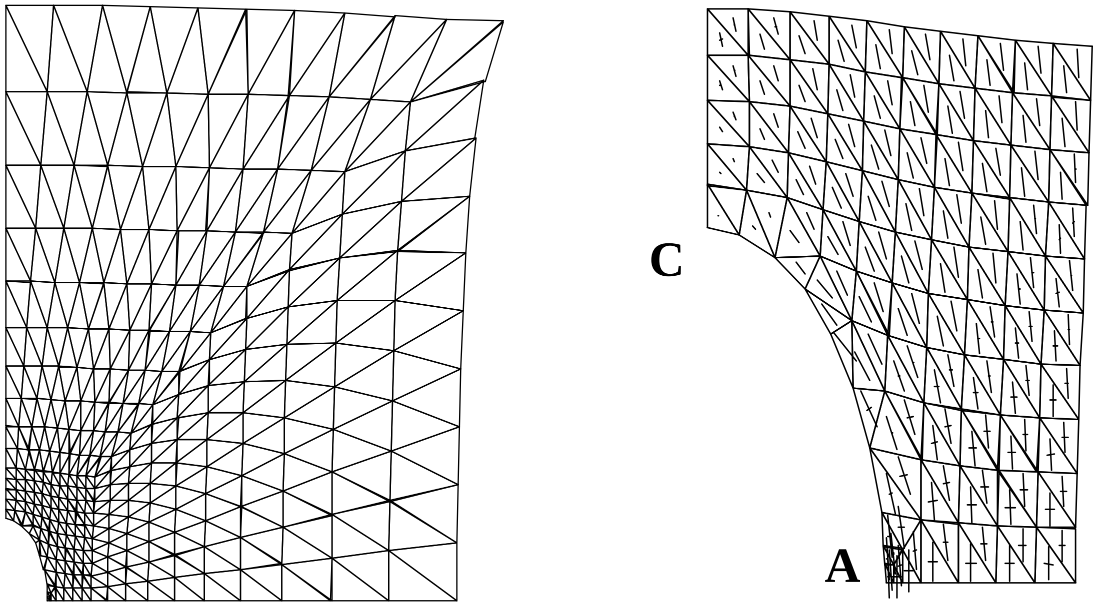
For the sake of comparison, the stated problem was firstly solved via the standard membrane theory and then by using the wrinkled membrane theory. Figure 2a shows the deformed shape obtained in the first case in correspondence to the load multiplier $\mu = 0.10$.



a) b)

Figure 2 – Membrane with a hole. Standard membrane theory: a) deformed shape; b) stress trajectories.

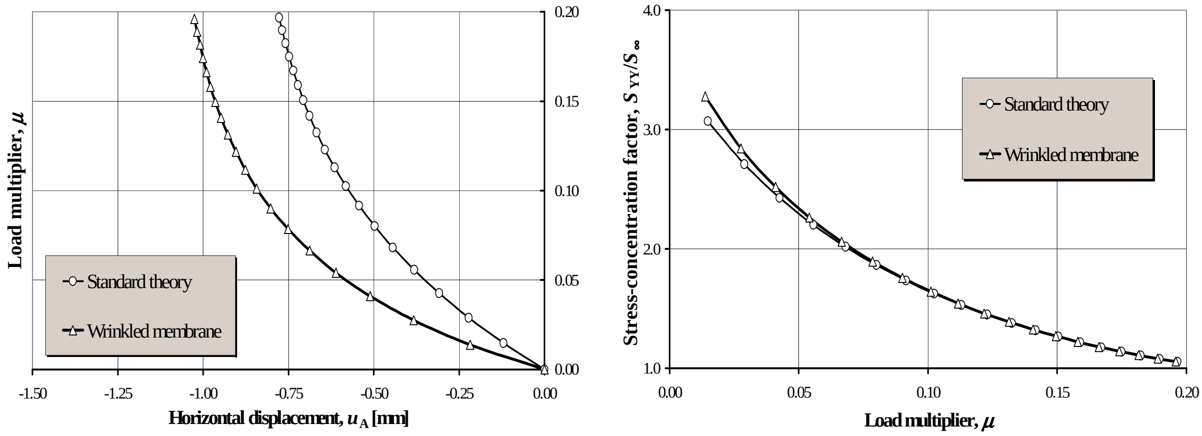
Figure 2b presents a detail of the principal stress trajectories in a small region surrounding the hole, where compressive stresses were made evident by thick lines.



a) b)

Figure 3 – Membrane with a hole. Wrinkled membrane theory: a) deformed shape; b) stress trajectories.

Figures 3a and 3b show instead the deformed shape and a detail of the principal stress trajectories, as furnished by the wrinkled membrane theory, for the same load multiplier. It is evident how compressive stresses are no more present. According to the above definitions, a small region appears over point **C**, which can be classified as a slack region, i.e., its elements are stress free. Moreover, adjacent elements appear to be tensioned along one principal direction and contracted along the other, thus giving rise to a contouring wrinkled region.



a) b)

Figure 4 – Membrane with a hole: a) equilibrium paths; b) stress concentration factors.

In Figure 4a the relating equilibrium paths are plotted in the $u_{\mathbf{A}}$-μ-plane, where $u_{\mathbf{A}}$ is the horizontal displacement of the point $\mathbf{A} \equiv [a,0]^{\mathrm{T}}$. Figure 4b shows instead the laws of variation of the ratio S_{YY} / S_{∞} between the normal component of stress $S_{YY}(a,0)$ and the reference load

S_∞. The two models show appreciable differences in terms of displacements rather then in stress, while a marked decrease of the stress-concentration factor is common to both cases.

5. A SQUARE MEMBRANE WITH A CENTRAL SLIT

Next, we consider a square membrane with a central slit (Figure 5a). The gross dimensions of the membrane and the boundary conditions are as before. The slit size is $2a = 10\,mm$.

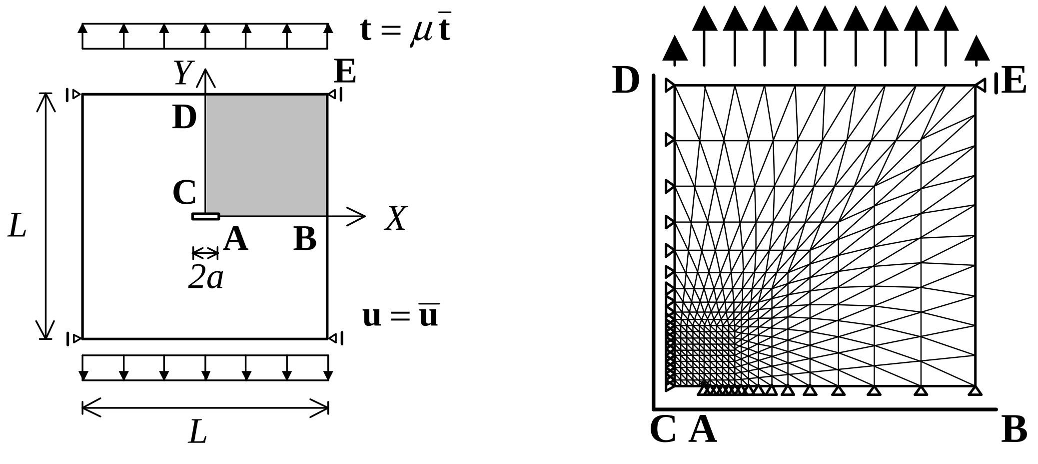


Figure 5 – A square membrane with a central slit: a) geometry and loads; b) FEM model.

Figure 5b illustrates the adopted FEM model, which comprises 341 nodes and 616 constant strain/constant stress triangular elements. The parameter values given in (14) were adopted.

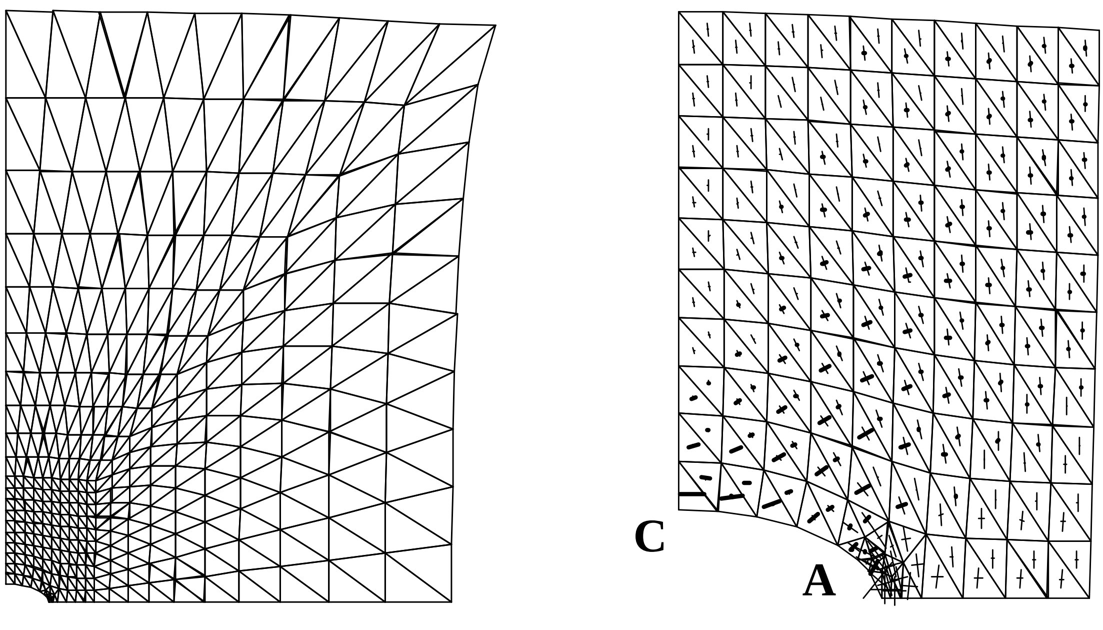
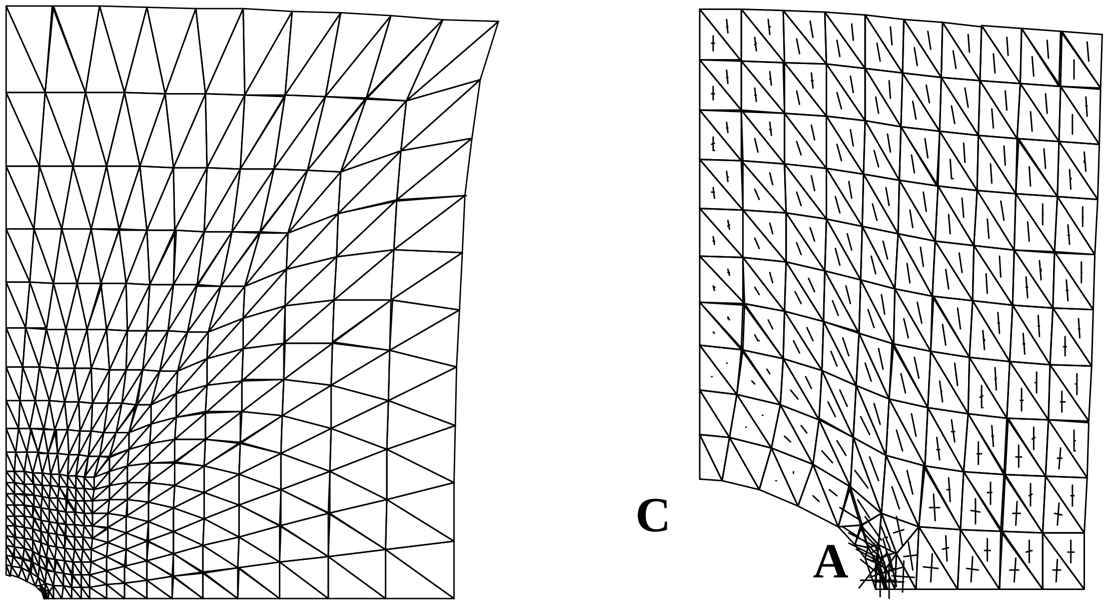


Figure 6 – Membrane with a slit. Standard membrane theory: a) deformed shape; b) stress trajectories.

Again, the stated problem was firstly solved via the standard membrane theory and then by using the wrinkled membrane theory. With reference to the first case, Figure 6a depicts the deformed shape for the load multiplier $\mu = 0.10$, while a detail of the principal stress trajectories in the region surrounding the slit is given in Figure 6b. Existing compressive stresses were made more evident by thick lines.

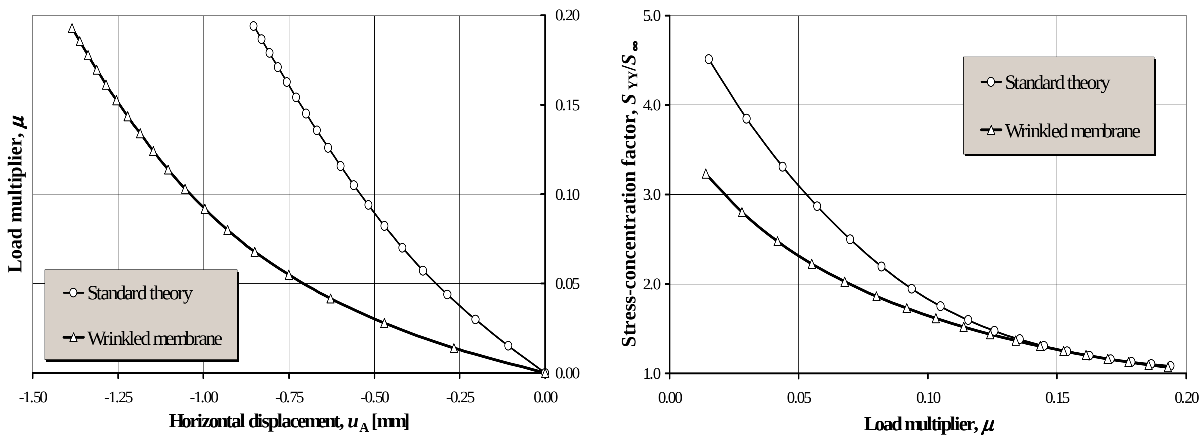
Figures 7a and 7b show instead the deformed shape and the detail of the principal stress trajectories obtained via the wrinkled membrane theory, for the same load multiplier.



a) b)

Figure 7 – Membrane with a slit. Wrinkled membrane theory: a) deformed shape; b) stress trajectories.

Finally, in Figure 8a the resulting equilibrium paths were plotted in the $u_{\mathbf{A}}$-μ-plane. Figure 8b shows the laws of variation of the stress ratio S_{YY} / S_∞. Again, a marked decrease of the stress-concentration factor is evident, but now the differences in terms of stress between the two solutions are no more negligible.



a) b)

Figure 8 – Membrane with a slit: a) equilibrium paths; b) stress concentration factors.

6. CONCLUSION

A model was presented for analysing the stress distribution, which arises in the neighbourhood of geometric discontinuities, such as holes or slits, in soft elastic membranes subject to increasing loads. The main sources of non-linearity present were taken simultaneously into account by recourse to specific tools. In particular, large displacements and large deformations were considered via the Green-Lagrange measure of strain, while material non-linearity was entered via Ogden's constitutive law. Moreover, local buckling or wrinkling phenomena were taken into account through the concept of *relaxed energy* proposed by Pipkin. The equilibrium problem was solved by recourse to the stationary total potential energy principle. Finally, numerical solutions for increasing load values were obtained by using a continuation procedure of the *arc-length* type, which allowed the tracing of the equilibrium path.

The proposed strategy features some not negligible innovations with respect to what is usually considered when modelling soft biological membranes ([6], [7]). In particular, three main aspects of novelty can be highlighted:

a) the peculiar property of biological materials of not being able to sustain any compressive stress is taken into account;
b) interaction between geometrical discontinuities and local buckling and wrinkling phenomena is fully considered;
c) the possibility of monitoring the evolution of the whole phenomenon for increasing load. In this case, for relevant quantities, such as stress-concentration factors and characteristic displacements, the laws of variation with the load multiplier were obtained.

To our opinion, these aspects can in many cases more relevantly affect the stress distribution, with respect to other ones, such as time dependency, anisotropy, etc., usually entered in more sophisticated constitutive laws.

Application of the resulting model to simple cases of rectangular membranes endowed with a central hole or a slit under uniaxial traction was satisfactory. In particular, a more realistic estimate of the stress concentration factors with respect to standard membrane theory was obtained. It is noteworthy in both cases how, after a very short initial stage of the loading process, the concentration factors resulted quite different and less severe from the values predicted from the Linear Elastic Fracture Mechanics.

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