On the problem of decomposing grasp and manipulation forces in multiple whole-limb manipulation

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23/05/2011

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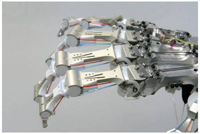


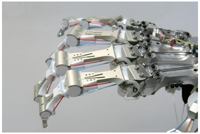
MANIPULATION

GRASP FORCES DECOMPOSITION

- Form Closure
- Force Closure
  - Power Grasp

SYSTEMS STRUCTURE

MANIPULATION



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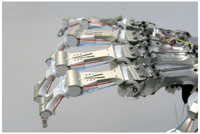

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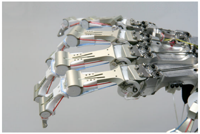

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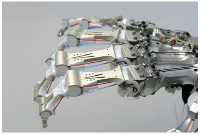

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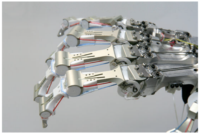

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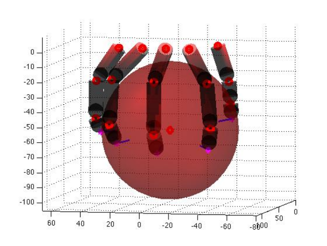
PROBLEM DEFINITION

Problem

Decomposing the system of contact forces exerted between the robot limbs and the object in order to apply a desired resultant force on the object (and/or to resist external disturbances)

BACKGROUND DEFINITIONS

- The problem of controlling contact forces in a multiple manipulation system such as a hand, a pair of cooperating robot arms, or a legged vehicle, has been traditionally considered in the assumption that every single finger (arm, or leg) has full mobility in its task space



$$w = -\tilde{G}\tilde{t}$$

$$\Downarrow$$

$$\tilde{t} = \tilde{G}^R w - Ax$$

$$\begin{cases} w = (f^T, m^T)^T \in \mathbb{R}^6 \\ G \in \mathbb{R}^{6\times t} \\ A \in \mathbb{R}^{t\times h} \end{cases}$$

BACKGROUND DEFINITIONS

Non defective systems

- Most known grasp optimization techniques can be formulated by defining a cost function $V(x)$ and constraint functions $g_i(x)$ as

  Find $\hat{x}$ *such that*
  $V(\hat{x}; w)$ *is minimum;*
  $gi(\hat{x}) \leq 0$

  1. The cost and constraint functions usually are designed so as to realize the goals of avoiding contact slippage and minimizing consumption of power in the joint actuators
  2. Finally $\hat{t}$ is applied by the fingers under some type of force control technique

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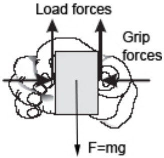
DEFECTIVE SYSTEMS

Fundamental question

What internal forces at equilibrium are modifiable at will, when inputs are joint torques?



DEFECTIVE SYSTEMS

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Definition

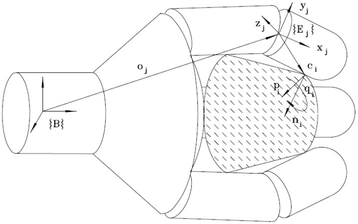
Grasping systems (or situation) where there is no guarantee that the optimal contact forces can actually be realized by the robot. In other words, complete (output function) controllability of internal forces may not be achieved in those cases.

- Quasi-static analysis $\Rightarrow$ to answer the fundamental question

MODELING THE SYSTEM

Preliminary assumptions

1. The model of the cooperating manipulation system we assume is comprised of
   - an **arbitrary number of robot fingers** (i.e., simple chains of links connected through revolute or prismatic joints)
   - **an object**, which is in contact with some or all of the links
2. We assume that **the location of the contact point is known**, by either planning or sensing



According to standard conventions

- Fixed base reference frame B
- local reference frames E_j, fixed to the j-th robot link
  - origin of E_j is placed on the j-th joint axis
  - z-axis of E_j is aligned with the j-th joint axis
  - x-axis of E_j is aligned with the line joining o_j with o_{j+1}
- All vectors are expressed in base frame unless explicitly noted

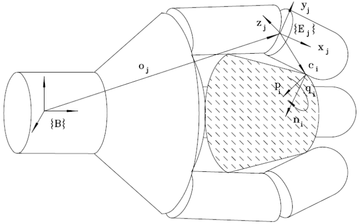
Object balance

- Contact wrench

  $\tilde{t} = (p_1^T, \dots, p_n^T, m_1^T \dots m_n^T)^T$

  $\tilde{t} \in \mathbb{R}^n$

- Balance equation



- Grasp matrix

- $S(c_i)$ is the cross-product matrix for c_i, hence the skew-symmetric matrix such that $S(c_i)p_i = c_i \times p_i$

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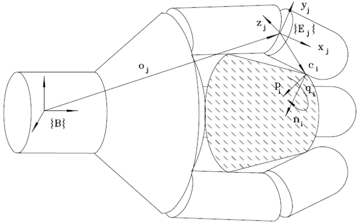
$$w = -\tilde{G}\tilde{t}$$



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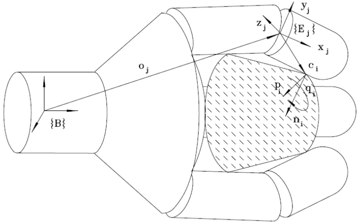

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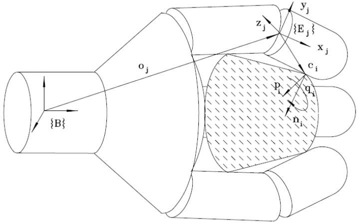

- Grasp matrix

$$\tilde{G} = \left(\begin{array}{cccc|cccc} I_3 & I_3 & \cdots & I_3 & & 0_{3\times 3n} & & \\ S(c_1) & S(c_2) & \cdots & S(c_n) & I_3 & I_3 & \cdots & I_3 \end{array} \right)$$

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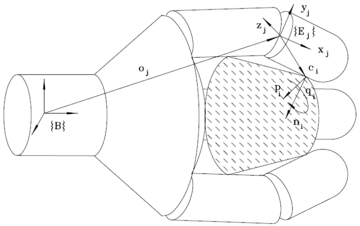

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$$\tilde{G} = \left(\begin{array}{cccc|cccc} I_3 & I_3 & \ldots & I_3 & O_{3\times 3n} & & & \\ S(c_1) & S(c_2) & \ldots & S(c_n) & I_3 & I_3 & \ldots & I_3 \end{array} \right)$$

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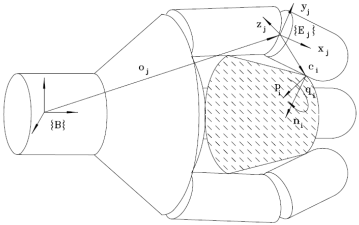

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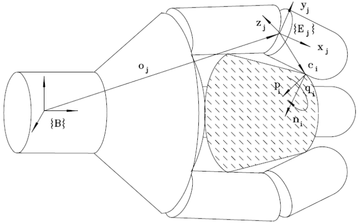

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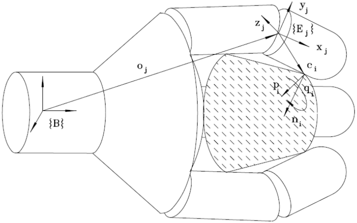

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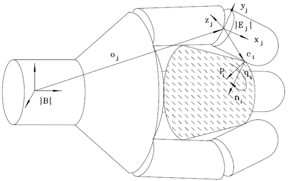
- $S(c_i)$ is the cross-product matrix for c_i, hence the skew-symmetric matrix such that $S(c_i)p_i = c_i \times p_i$

Manipulation system balance

$$\tau = \tilde{J}^T \tilde{t}; \qquad \tilde{J} \in \mathbb{R}^{6n \times q}$$



Defective systems

$$q \geq rank(J) < t$$

J matrix

$$\tilde{J}^T = \left(\begin{array}{cccc|cccc} D_{1,1} & D_{2,1} & \dots & D_{n,1} & L_{1,1} & L_{2,1} & \dots & L_{n,1} \\ D_{1,2} & D_{2,2} & \dots & D_{n,2} & L_{1,2} & L_{2,2} & \dots & L_{n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{1,n} & D_{2,n} & \dots & D_{n,n} & L_{1,n} & L_{2,n} & \dots & L_{n,n} \end{array} \right)$$

$$D_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact force does not affect the } j\text{-th joint;} \\ z_j^T & \text{for prismatic } j\text{-th joint;} \\ z_j^T S(c_i - o_j) & \text{for revolute } j\text{-th joint;} \end{cases}$$

$$L_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact torque does not affect the } j\text{-th joint;} \\ O_{1\times 3} & \text{for prismatic } j\text{-th joint;} \\ z_j^T & \text{for revolute } j\text{-th joint;} \end{cases}$$

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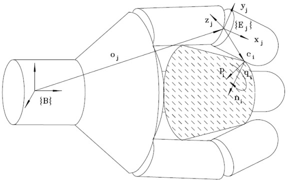

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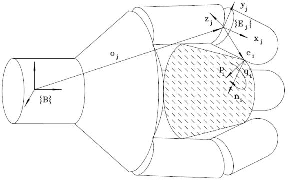

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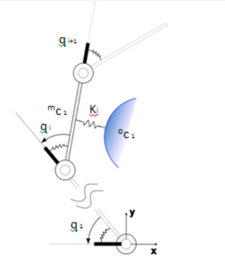
Contact constraint

To incorporate contact constraints in the model, **relative displacements between the object and the links at the contact points must be considered**. Therefore, we introduce n reference frames ${}^{o}C_i$ fixed w.r.t. the object and centered in c_i ; and n reference frames ${}^{m}C_i$, each fixed w.r.t. the link that touches the object in c_i, and centered in c_i

$$\Delta^{o}x = \tilde{G}^{T}\Delta u$$

$$\Delta^{m}x = \tilde{J}\Delta q$$



Contact type

Contact constraints impose that certain components of the relative displacements $\Delta^o x - \Delta^m x$ are selectively opposed by reaction forces, depending upon the type of contact:

1. Complete constraint
2. Soft finger
3. Hard finger w/h friction
4. Hard finger w/o friction

Imposing constraints

$$H(\Delta^m x - \Delta^o x) = 0$$

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Selection Matrix H

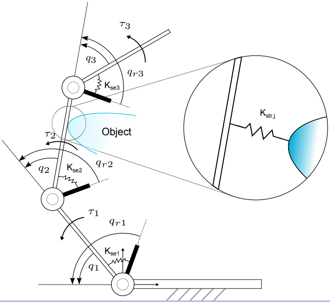
The selection matrix H is build by removing all the zero rows from :

$$\tilde{H} = \mathrm{diag}(FS_1, \ldots, FS_n, MS_1, \ldots, MS_n)$$

Contact Type	Force Selector FS_i	Moment Selector MS_i
Point Contact w/o Friction	z_i^T	$O_{1\times(d-s)}$
Point Contact w/h Friction (hard-finger)	I_s	$O_{1\times(d-s)}$
Planar Contact w/o Friction (Complete-Constraint)	I_s	I_{d-s}
3D Soft Finger	I_3	z_i^T

Virtual spring

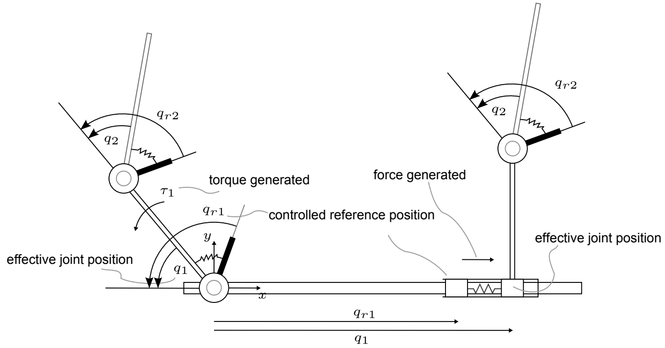
All relationship considered so far are valid for a **rigid-body model** of the robot system. However, the **force distribution problem for general systems is underdetermined**. To solve the indeterminacy, the rigid body model is inadequate, and a more accurate model, taking into account the **elastic elements that are involved in the system**, has therefore to be considered. This can be conceptually done by introducing a set of "virtual springs" interposed between the links and the object at the contact points



Elastic relationship

$C_{str} t = H(\Delta^m x - \Delta^o x) + t_0$

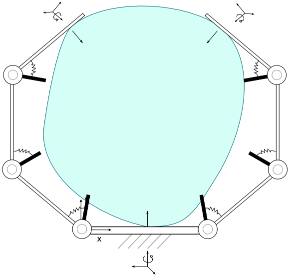
$t \in \mathbb{R}^t$ differs from $\tilde{t} = H^T t \in \mathbb{R}^{6n}$



Joint actuation-control model

$$C_{se}\tau = (q_r - q)$$

From the above presented equations it holds

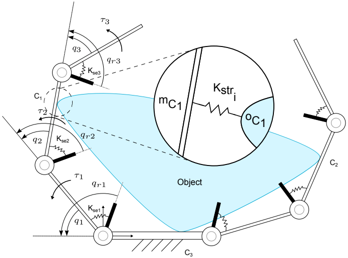


$$
\begin{cases}
w = -\tilde{G}\tilde{t} \\
\tau = \tilde{J}^T \tilde{t} \\
C_{str} t = H(\Delta^m x - \Delta^o x) + t_0 \\
\tilde{t} = H^T t \\
\Delta^o x = \tilde{G}^T \Delta u \\
\Delta^m x = \tilde{J}^T \Delta q \\
C_{se} \tau = (q_r - q)
\end{cases}
$$

QUASI-STATIC MODEL

Defining

$$G = \tilde{G}H^T$$
$$J = H\tilde{J}$$
$$\tilde{t} = H^T t$$



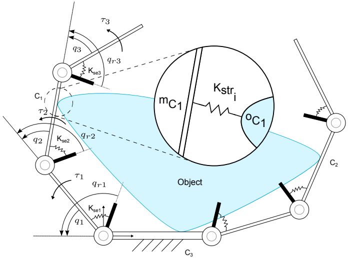
"To study" model

$$\begin{cases} w = -Gt \\ \tau = J^T t \\ C_{str} t = (J\Delta q - G^T \Delta u) \\ C_{se}\tau = (q_r - q) \end{cases}$$

QUASI-STATIC MODEL

Defining

$$G = \tilde{G} H^T$$
$$J = H \tilde{J}$$
$$\tilde{t} = H^T t$$



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GRASP FORCES DECOMPOSITION

Fundamental question

What internal forces at equilibrium are modifiable at will, when inputs are joint torques?

$$\tilde{t} = \tilde{G}^R w - Ax \Longrightarrow \begin{cases} \tilde{t}_{ps} = \tilde{G}^R w \\ \tilde{t}_{omo} = Ax \end{cases}$$

The stiffness matrix K

$$\begin{cases} w = -Gt \\ \tau = J^T t \\ C_{str} t = (J\Delta q - G^T \Delta u) \\ C_{se}\tau = (q_r - q) \end{cases}$$

$$\begin{cases} \delta\tau = J^T \delta t \\ C_{se}\delta\tau = (\delta q_r - \delta q) \end{cases} \Rightarrow C_{se} J^T \delta t = (\delta q_r - \delta q)$$

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Stiffness matrix

$$K = [C_{str} + JC_{se}J^T]^{-1}$$

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Stiffness matrix

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Current system

$$\begin{cases} \delta w = -G\delta t \\ \delta \tau = J^T \delta t \\ \delta t = K(J\delta q_r - G^T \delta u) \end{cases}$$

PARTICULAR SOLUTION

The particular solution $\tilde{t}_{ps} = \tilde{G}^R w$ is not unique, since G in general admits infinitely many right inverses. However, we expect a unique solution to the following *Force distribution problem*.

Force distribution problem

Assume that an object, at equilibrium under an external load w_0 and contact forces t_0, is subject to an additional load w, while all other parameters are kept constant.
Determine the values of contact forces at the new equilibrium.

Force distribution problem solution

The solution to the force distribution problem is unique, and is given by

$$t = G_K^R w + t_0$$

where $G_K^R = KG^T(GKG^T)^{-1}$

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HOMOGENEOUS SOLUTION

Internal forces are self-balanced contact forces that have no effect on the global motion of the manipulated object but significantly affect the grasp stability, have been identified with homogeneous solutions of $w = -Gt$. In mathematical terms, internal forces are elements of the subspace $N(G)$ and hence $R(A)$. We propose a decomposition of the homogeneous subspace in a subspace F_{hr} of active, internal contact forces and a subspace F_{ho} of passive (preload), internal contact forces.

Internal forces

$$\text{Internal forces} \Longrightarrow \begin{cases} \text{Active internal forces} \\ \text{Passive internal forces} \end{cases}$$

Active internal Forces

$$t = Ey$$

where $E = (I - G_K^R G)KJ$

Passive internal forces

$$t = Pz$$

where $P = R(A) \cap N(J^T)$

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$$t = -KG^T \Delta u + t_0$$

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- Now assuming G full row rank and K invertible

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Active internal forces

Consider a system in the equilibrium configuration described by w_0, q_0, to, and let δu be a displacement of the object which is compatible with all the constraints imposed by contacts with the robot links (i.e., δu is a virtual displacement of the object). Applying the P.V.W.

Principle of Virtual Work

The principle that the total work done by all real/virtual forces acting on a system in static equilibrium is zero for any virtual/real displacement from equilibrium which is consistent with the constraints of the system.

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...applying the P.V.M.

$$w_0^T \delta u = t_0^T G^T \delta u = 0, \quad \forall \delta u$$

- By imposing joint displacements Δq_r, the equilibrium configuration is perturbed. A new equilibrium under the same external force w_0 will be reached on condition that the P.V.W. is satisfied

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$$GKJ\Delta q_r = GKG^T \Delta u \Rightarrow \Delta u = (GKG^T)^{-1} GKJ\Delta q_r$$

Active internal forces

$\Delta t = (I - G_K^R G)KJ\Delta q_r$ and so

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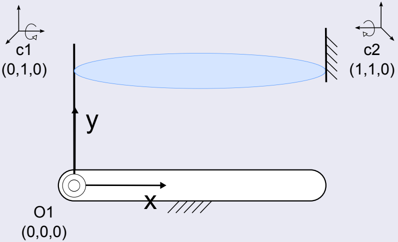
A simple, simple example

Structure

Let 's examine the example reported below



We have to build:

- G matrix
- J matrix
- H matrix
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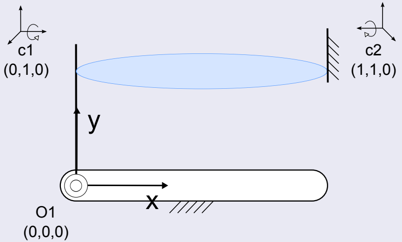

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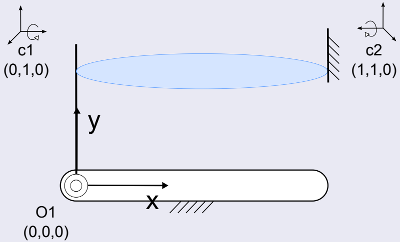

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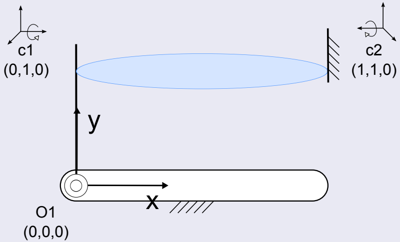

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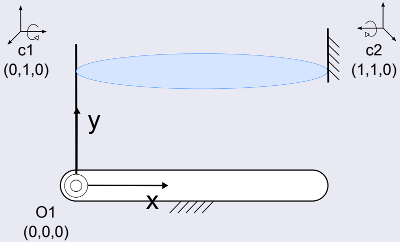

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G matrix

$$\tilde{G} = \left(\begin{array}{cccc|cccc} I_3 & I_3 & \dots & I_3 & O_{3\times 3n} & & & \\ S(c_1) & S(c_2) & \dots & S(c_n) & I_3 & I_3 & \dots & I_3 \end{array} \right)$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & -1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

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A simple, simple example

J matrix

$$\tilde{J}^T = \left(\begin{array}{cccc|cccc} D_{1,1} & D_{2,1} & \dots & D_{n,1} & L_{1,1} & L_{2,1} & \dots & L_{n,1} \\ D_{1,2} & D_{2,2} & \dots & D_{n,2} & L_{1,2} & L_{2,2} & \dots & L_{n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{1,n} & D_{2,n} & \dots & D_{n,n} & L_{1,n} & L_{2,n} & \dots & L_{n,n} \end{array}\right)$$

$$D_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact force does not affect the } j\text{-th joint;} \\ z_j^T & \text{for prismatic } j\text{-th joint;} \\ z_j^T S(c_i - o_j) & \text{for revolute } j\text{-th joint;} \end{cases}$$

$$L_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact torque does not affect the } j\text{-th joint;} \\ O_{1\times 3} & \text{for prismatic } j\text{-th joint;} \\ z_j^T & \text{for revolute } j\text{-th joint;} \end{cases}$$

$$J^T = [\ -1.0\ \ 0.0\ \ 0.0\ \ 0.0\ \ 0.0\ \ 0.0\ \ 0.0\ \ 0.0\ \ 1.0\ \ 0.0\ \ 0.0\ \ 0.0\ ]$$

A simple, simple example

J matrix

$$\tilde{J}^T = \left(\begin{array}{cccc|cccc} D_{1,1} & D_{2,1} & \dots & D_{n,1} & L_{1,1} & L_{2,1} & \dots & L_{n,1} \\ D_{1,2} & D_{2,2} & \dots & D_{n,2} & L_{1,2} & L_{2,2} & \dots & L_{n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{1,n} & D_{2,n} & \dots & D_{n,n} & L_{1,n} & L_{2,n} & \dots & L_{n,n} \end{array}\right)$$

$$D_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact force does not affect the } j\text{-th joint;} \\ z_j^T & \text{for prismatic } j\text{-th joint;} \\ z_j^T S(c_i - o_j) & \text{for revolute } j\text{-th joint;} \end{cases}$$

$$L_{i,j} = \begin{cases} O_{1\times 3} & \text{if the } i\text{-th contact torque does not affect the } j\text{-th joint;} \\ O_{1\times 3} & \text{for prismatic } j\text{-th joint;} \\ z_j^T & \text{for revolute } j\text{-th joint;} \end{cases}$$

$$J^T = \begin{bmatrix} -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

A simple, simple example

Selection Matrix H

The selection matrix H is build by removing all the zero rows from :

$$\tilde{H} = \mathrm{diag}(FS_1, \ldots, FS_n, MS_1, \ldots, MS_n)$$

Contact Type	Force Selector FS_i	Moment Selector MS_i
Point Contact w/o Friction	z_i^T	$O_{1 \times (d-s)}$
Point Contact w/h Friction (hard-finger)	I_s	$O_{1 \times (d-s)}$
Planar Contact w/o Friction (Complete-Constraint)	I_s	I_{d-s}
3D Soft Finger	I_3	z_i^T

A simple, simple example

$$H = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 \end{bmatrix}$$

A simple, simple example

K matrix

$$C_{str} = \begin{bmatrix} 0.05 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.05 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.05 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.05 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.05 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.05 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.01 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.01 \end{bmatrix} \qquad C_{ser} = 0.01$$

K matrix

$$K = [C_{str} + JC_{se}J^T]^{-1} = \begin{bmatrix} 20.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 20.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 20.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 20.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 20.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 20.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 100.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 100.0 \end{bmatrix}$$

Forces decomposition subspaces

Particular solution

$$PS = \begin{bmatrix} -0.45455 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & -1.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 \\ -0.54545 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.50000 & -0.50000 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.50000 & 0.50000 & 0.0 & 0.0 \end{bmatrix}$$

Forces decomposition subspaces

Homogeneous solution

$$E = \begin{bmatrix} 0.5 \\ 0.0 \\ 0.0 \\ -0.5 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} \qquad P = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.3 \\ 0.3 \end{bmatrix}$$

Grazie per l'attenzione..

Reference

[1] Antonio Bicchi. "On the problem of decomposing grasp and manipulation forces in multiple whole limb manipulation". In: *Int. Journal of Robotics and Autonomous Systems vol. 13 (1994), pp. 127–147.Elsevier Science, Oxford, UK.* 3 (1994).

SKEW MATRIX

Alternative ways to compute the cross product

Conversion to matrix multiplication

The vector cross product also can be expressed as the product of a skew-symmetric matrix and a vector:[8]

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{b}]_{\times}^{\top} \mathbf{a} = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where superscript $^{\top}$ refers to the Transpose matrix, and $[\mathbf{a}]_{\times}$ is defined by:

$$[\mathbf{a}]_{\times} \stackrel{\mathrm{def}}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Also, if $\mathbf{a}$ is itself a cross product:

$$\mathbf{a} = \mathbf{c} \times \mathbf{d}$$

then[*citation needed*]

$$[\mathbf{a}]_{\times} = (\mathbf{c}\mathbf{d}^{\top})^{\top} - \mathbf{c}\mathbf{d}^{\top}.$$