State-based models

State-based models

"Reliability depends on the frequency of faults and the duration of faults in the system"

Series/Parallel models: relate the reliability of the system to the system structure and to the reliability of its components. If there is a path from the input node to the output node the system behaves correctly. If there is a failed component on a path, the path is broken. Duration of faults is not considered.

State-based models: enumerate the system states. Can be used for Reliability and Availability measures

Definition of dependability attributes

Reliability - R(t)

conditional probability that the system performs correctly throughout the interval of time [t0, t], given that the system was performing correctly at the instant of time t0

```
 |-------------------------------|--    no failures in the interval [t0, t]
t0                              t
```

Availability - A(t)

the probability that the system is operating correctly and is available to perform its functions at the instant of time t

```
 |-------------XXXXX--------------XXX-------------------------|--   failures and repairs
t0            [t1….t2]          [t3…t4]                       t     in  the interval [t0, t]
               f     r           f    r
```

State-based models

Characterize the state of the system changing over time

1. Each state represents a distinct combination of failed and working modules

2. The system goes from state to state as modules fail and repair

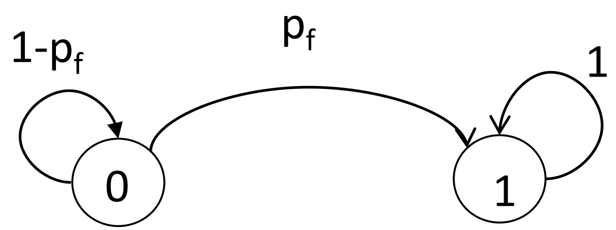
3. The state transitions are characterized by the probability of failure and the probability of repair

4. The time between a fault and a repair is the duration of the fault inside the system

State-based models

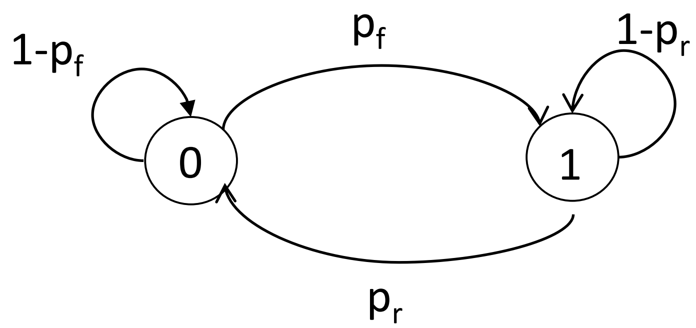
graph where nodes are all the possible states and arcs are the possible transitions between states. Arcs are labeled with a probability function

Example:
System that consists of one module



Reliability model

The module can be
- working (state 0)
- faulty (state 1)



Availability model

- pf: probability of failure
- pr: probability of repair

Random process

In probability theory, a **random process** is defined as a family of random variables indexed by numbers expressing points in time

Example of random process $\{X_t\}$, with time $t = \{0, 1, 2, 3. ...\}$

Let X be the result of tossing a die.
$\{X_t\}$ represents the sequence of results of tossing a die

$P[X_0 = 4] = 1/6$
$P[X_4 = 4 \mid X_2 = 2] = P[X_4 = 4\ ] = 1/6$

$S=\{1,2,3,4,5,6\}$ state space

P probability

Dependability measures: these variables represent the values of the state of the system randomly changing over time

Markov process

In a general random process $\{X_t\}$, the value of the random variable X_{t+1} may depend on the values of the previous random variables

$$X_0 \, X_1 \ldots\ldots\ldots\ldots X_t$$

Markov process

the state of the process at time t+1 depends only on the state at time t, and is independent on any state before t

$$P\{X_{t+1}=j \;|\; X_1=j_1, \ldots, X_{t-1}=j_{i-1}, \ldots, X_t=i\} = P\{X_{t+1}=j \;|\; X_t=i\}$$

Markov property: “the current state is enough to determine the future state”

State: number working and faulty modules

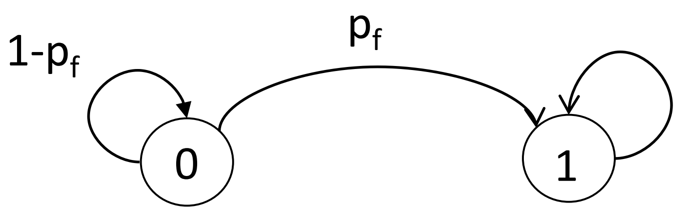
Example: Simplex system

$\{X_t\}$ t=0, 1, 2, …. S={0, 1}

- all state transitions occur at fixed intervals
- probabilities assigned to each transition
- the probability of state transition depends only on the current state

State 0 : working
State 1: failed

p_f Failure probability



Transition probability matrix P (nxn)

- pij = probability of a transition from state i to state j
- pij >=0
- the sum of each row must be one

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-p_f & p_f \\ 0 & 1 \end{bmatrix} \end{array}$$

next state

current state

Markov process

Markov process (a special type of random process) :

Basic assumption: the system behavior at any time instant depends only on the current state (independent of past values)

Steady-state transition probability: the probability of transition from state i to state j does not depend by the time.

$$P\ \{X_1=j \mid X_0=i\} = P\ \{X_2=j \mid X_1=i\} = \ldots = P\ \{X_{t+1}=j \mid X_t=i\}$$

This probability is called p_{ij}

$$p_{i\,j} = P\ \{X_1=j \mid X_0=i\}$$

Transition probability matrix

if a Markov process is finite-state, and has *steady-state transition probabilities* we can define the transition probability matrix P (nxn)

$$P = \begin{array}{cc} & \begin{array}{ccc} \phantom{p_{11}} & j & \phantom{p_{1n}} \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \cdots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \end{array}$$

States are numbered starting from 1 to n in the matrix

pij = probability of moving from state i to state j in one step

row i of matrix P: probability of make a transition starting from state i

column j of matrix P: probability of making a transition from any state to state j

Markov chain

Markov chain: Markov process $\{X_t\}$ with discrete state space S

Homogeneous Markov chain: *steady-state transition probabilities*

We consider only *homogeneous* Markov chains

- discrete-time Markov chains (DTMC) / Continuous-time Markov chains (CTMC)

Discrete-time Markov chain (DTMC)

System evolution in a finite number of steps computed starting from the initial state occupancy vector and the transition probability matrix

States numbered starting from 0

State occupancy vector at time t $\quad \pi(t) = [\pi_0(t), \pi_1(t), \ldots]$

where $\pi_i(t)$ is the probability that $\{X_t = i\}$

Initial state occupancy vector $\quad \pi(0) = (\pi_0(0), \pi_1(0), \ldots)$

$\pi(1) = \pi(0)\,P$ state occupancy vector after one step

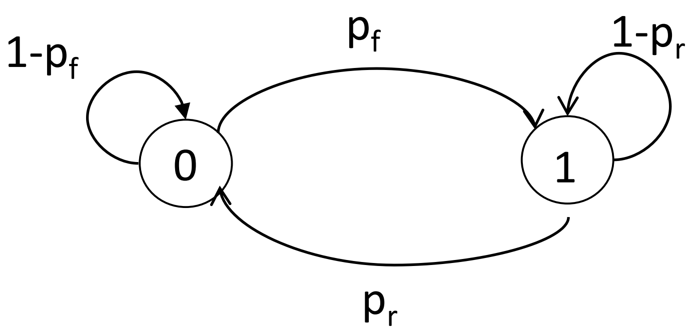
$\pi(t) = \pi(0)\,P^t$ state occupancy vector after t step

Simplex system with repair

State 0 : working
State 1: failed

p_f Failure probability

p_r Repair probability

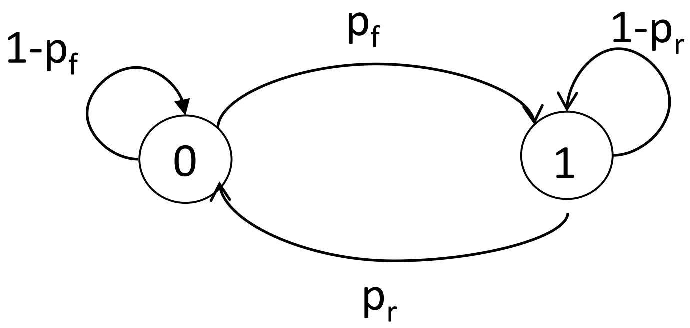


$$
\begin{array}{c c c l}
 & 0 & 1 & \text{next state} \\
P = \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{c} 1-p_f \\ p_r \end{array}\right. & \left.\begin{array}{c} p_f \\ 1-p_r \end{array}\right] & \\
\text{current state} & & &
\end{array}
$$

Simplex system with repair

$$\pi(0) = (\pi_0(0), \pi_1(0))$$

probability of being in a state at the beginning



$$\pi(1) = \pi(0) \begin{bmatrix} 1-p_f & p_f \\ p_r & 1-p_r \end{bmatrix}$$

probability of being in a state after 1 time-step

$$\pi(t) = \pi(0) \begin{bmatrix} 1-p_f & p_f \\ p_r & 1-p_r \end{bmatrix}^t$$

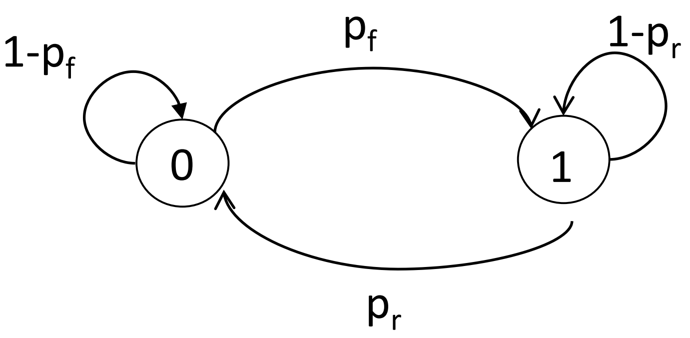
probability of being in a state after t time-steps

Simplex system with repair

$\pi(0) = [\ 1, 0]$ initial state: working

$$\pi(1) = [\ 1, 0] \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = [\ 0.9, 0.1]$$

probability of being in state 0
after 1 time-step is 0.9



$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

DTMC

$S=\{0, 1\}$
$\{X_t\}$ $t=0, 1,2$ ...

$X_0 = 0$ prob. 1
$X_0 = 1$ prob. 0

$X_1 = 0$ prob. 0.9
$X_1 = 1$ prob. 0.5
....

$X_t = 0$ prob that the system 's state is 0 at time t

Limiting behaviour

A Markov process can be specified in terms of the state occupancy vector π and the transition probability matrix P (transient behavior)

$$\pi(t) = \pi(0)\, P^t$$

The limiting behaviour of a DTMC (steady-state behaviour)

$$\lim_{t \to \infty} \pi(t)$$

The limiting behaviour of a DTMC depends on the characteristics of its states. Sometimes the solution is simple

Time-average state space distribution

For periodic Markov chains $\lim_{t \to \infty} \pi(t)$

doesn't exist (caused by the probability of the periodic state)

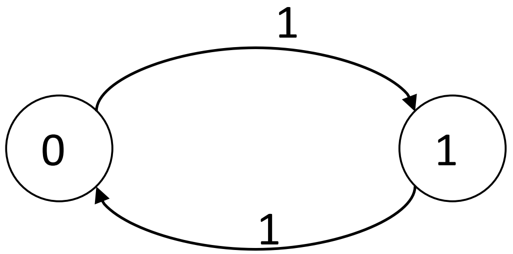
Initial state: working

$\pi(0) = (1,0)$

$\pi(1) = \pi(0)\, P = (1,0) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (0,1)$

$\pi(2) = \pi(1)\, P = (0,1) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (1,0)$

………..



$\pi(0) = (1,0)$

state i is periodic with period d=2

Compute the time-average state space distribution, sometimes called π^*

Continuous-time Markov model

- state transitions occur at random intervals
- transition rates assigned to each transition

Markov property assumption

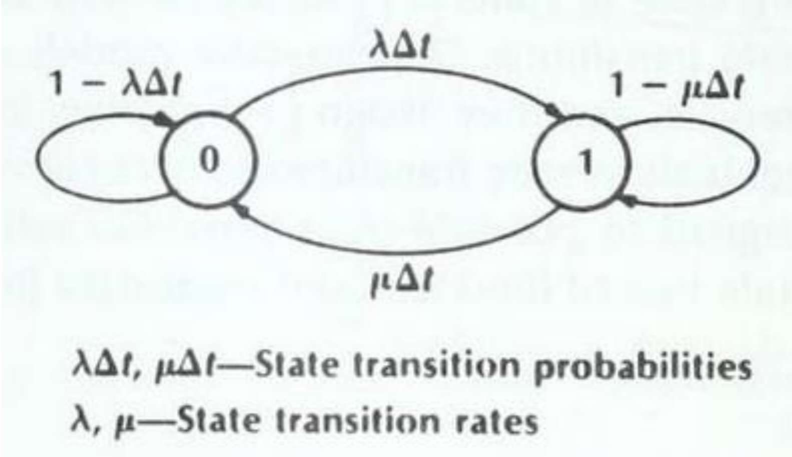
the length of time already spent in a state does not influence either the probability distribution of the next state or the probability distribution of remaining time in the same state before the next transition

These assumptions imply that the waiting time spent in any one state is exponentially distributed

Thus the Markov model naturally fits with the standard assumptions that failure rates are constant, leading to exponential distribution of interarrivals of failures

state 0: working
state 1: failed

λ failure rate
μ repair rate



Taken from: [Siewiorek et al.1998]

Continuous time

Transition matrix P: transition rate

Probability of being in state 0 or 1 at time t+Δt

$$P = \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

Simplex system with repair

$$\pi(t+\Delta t) = \pi(t) \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

Performing multiplication, rearranging and dividing by Δt, taking the limit as Δt approaches to 0:

$$\frac{d\pi_0(t)}{dt} = -\lambda\ \pi_0(t) + \mu\ \pi_1(t)$$

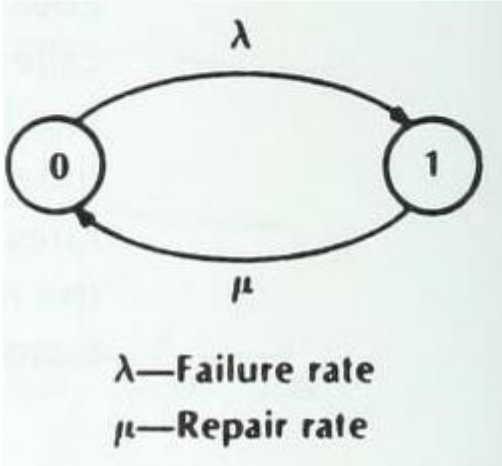
$$\frac{d\pi_1(t)}{dt} = \lambda\pi_0(t) - \mu\ \pi_1(t)$$

Simplex system with repair

$$\left[\frac{d\pi_0(t)}{dt}, \frac{d\pi_1(t)}{dt}\right] = \pi(t)\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Q matrix

Continuous time Markov model graph



Taken from: [Siewiorek et al.1998]

State 0: is minus the flow out of state 0 times the probability of being in state 0 at time t, plus the flow into state 0 from state 1 times the probability of being in state 1.

The set of equations can be written by inspection of a transition diagram without self-loops and Δt's

State-transition rate matrix Q

The matrix Q is defined as follows

$$q_{ij} = \begin{cases} \text{rate of going from state } i \text{ to state } j & \text{if } (i \mathrel{!=} j) \\ -\sum_{k != i} q_{ik} & \text{otherwise} \end{cases}$$

the sum of each row must be zero

A CTMC can be specified in terms of the occupancy probability vector π and the state-transition-rate matrix Q

Simplex system with repair

steady-state term

$$\pi_0(t) = \underbrace{\frac{\mu}{\lambda + \mu}}_{\text{steady-state term}} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$\pi_1(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

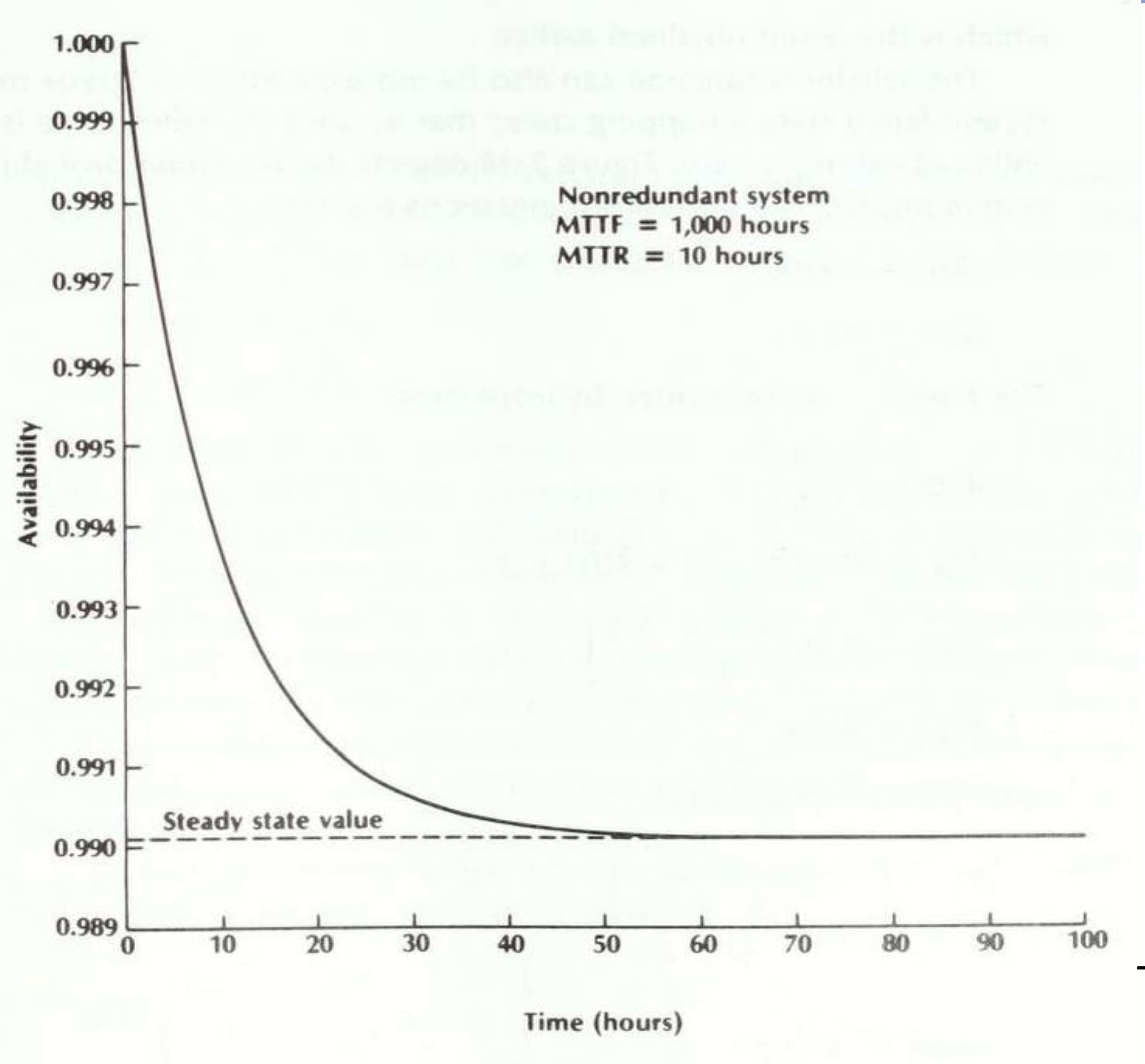
Taken from: [Siewiorek et al.1998]

$A(t) = \pi_0(t)$

$\pi_0(t)$ is the probability that the system is in the operational state at time t, availability at time t

The availability consists of a steady-state term and an exponential decaying transient term

Availability as a function of time



$\lambda = 0.001$ (1/1000)

$\mu = 0.1$ (1/10)

The steady-state value is reached in a very short time

Taken from: [Siewiorek et al.1998]

Continuous-time Markov models: Reliability

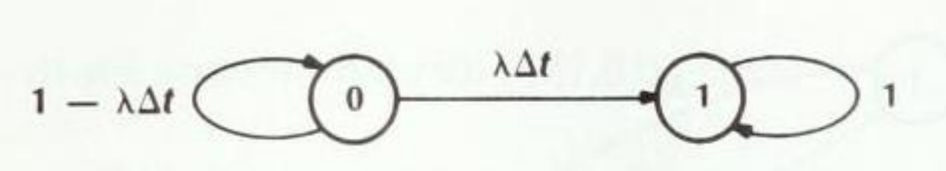
Single system without repair

failed state as trapping state

λ = failure rate

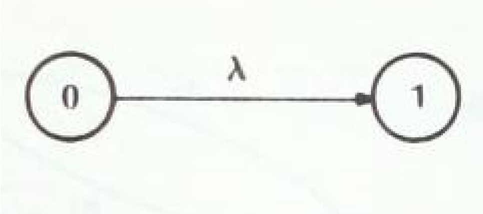
$\lambda\Delta t$ = state transition probability

$$Q = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$



Taken from: [Siewiorek et al.1998]

Continuous time Markov model graph



$$\pi_0(t) = e^{-\lambda t}$$ Reliability

$$\pi_1(t) = 1 - e^{-\lambda t}$$ Unreliability

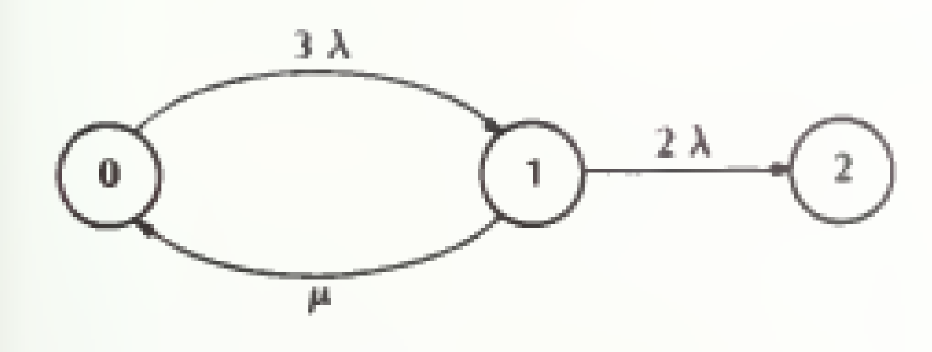
TMR system with repair

Rates: λ and μ

Identification of states:
- 3 processors working, 0 failed
- 2 processors working, 1 failed
- 1 processor working, 2 failed

$\pi(0) = [1, 0, 0]$

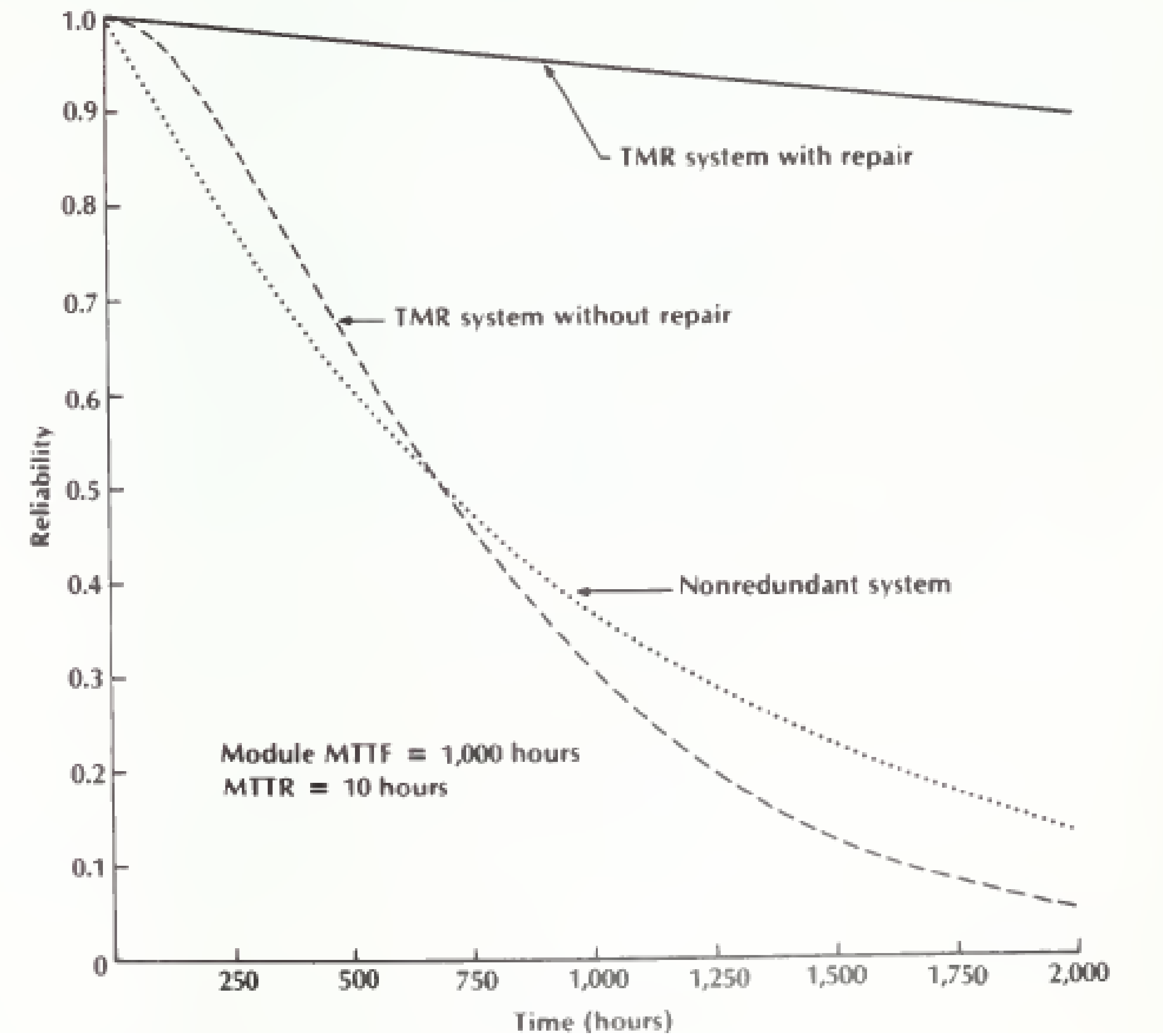
$$Q = \begin{bmatrix} -3\lambda & 3\lambda & 0 \\ \mu & -2\lambda-\mu & 2\lambda \\ 0 & 0 & 0 \end{bmatrix}$$



Taken from: [Siewiorek et al.1998]

Reliability $R(t) = 1 - \pi_2(t)$

Comparison with nonredundant system and TMR without/with repair



Taken from: [Siewiorek et al.1998]

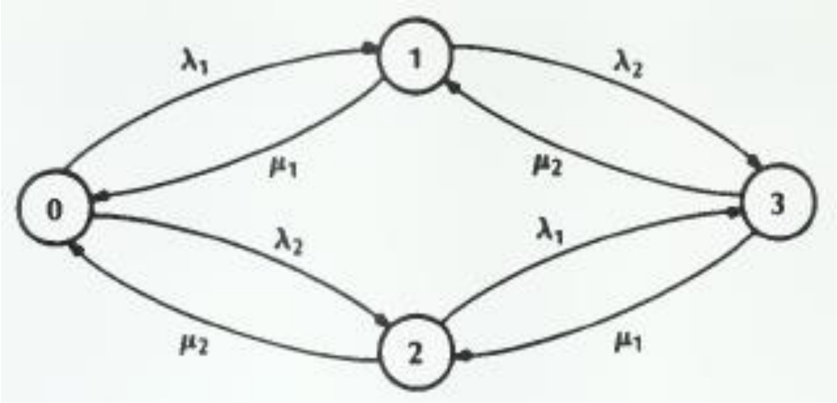
Dual processor system with repair

A, B processors Rates: $\lambda1$, $\lambda2$ and $\mu1$, $\mu2$

$\pi(0) = [1, 0, 0, 0]$

Identification of states:

- A, B working: state 0
- B working, A failed: state 1
- A working, B failed: state 2
- A, B failed: state 3



Availability

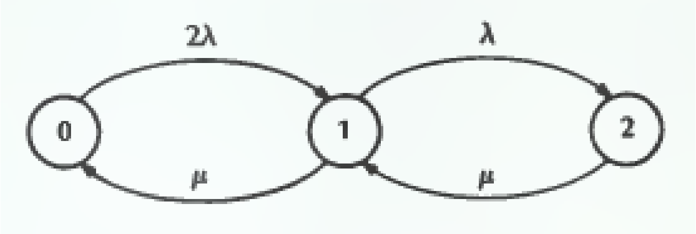
$A(t) = \pi_0(t) + \pi_1(t) + \pi_2(t)$

$A(t) = 1- \pi_3(t)(t)$

Rates: $\lambda1 = \lambda2$ and $\mu1 = \mu2$

$\pi(0) = [1, 0, 0]$

$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$



Taken from: [Siewiorek et al.1998]

Availability

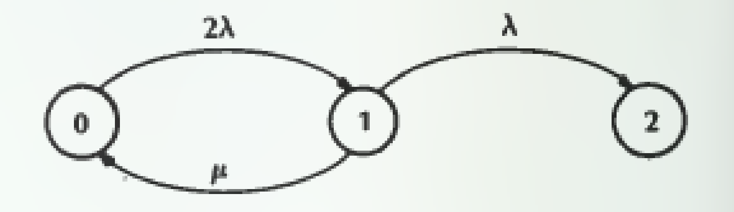
$A(t) = 1- \pi_2(t)$

Steady state value

$$A_{ss} = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2}$$

Dual processor system with repair: Reliability model

making state 2 a trapping state



Taken from: [Siewiorek et al.1998]

$\pi(0) = [1, 0, 0]$

$$T = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Reliability $R(t) = 1 - \pi_2(t)$

$R(t) = \pi_0(t) + \pi_1(t)$