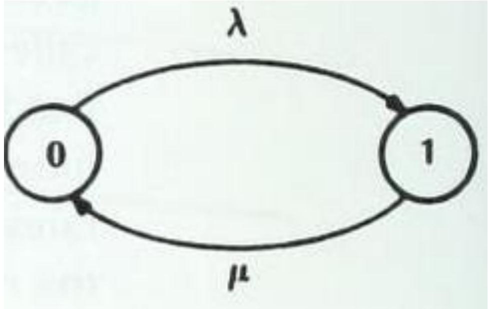
Continuous-time Markov chain: Single system with repair

- transition rates: λ failure rate, μ repair rate
  - identification of states
  - initial state-space p(0) = [1, 0]

$\pi^{(t)} = [\pi_0^{(t)}, \pi_1^{(t)}]$

$p(t) = [p_0(t), p_1(t)]$ in the book



$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - \mu p_1(t)$$

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Solution of the differential equations:

$$p_0(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$p_1(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

Availability

$A(t) = p_0(t)$

Continuous-time Markov chain (CTMC)

- Markov process $\{X_t\}$ with discrete-state space S
- steady-state transition rates
- events may occur at any point in time

  T is an interval of real numbers (>=0)

Memoryless property:

$$\mathcal{P}\{X_{t+\tau} = j | X_t = i, X_{t-t_1} = k_1, ..., X_{t-t_n} = k_n\} = \mathcal{P}\{X_{t+\tau} = j | X_t = i\}$$

for all τ>0 and 0<t1<t2<…<tn.

Steady-state transition probabilities

$$\mathcal{P}\{X_{t+\tau} = j | X_t = i\} = \mathcal{P}\{X_\tau = j | X_0 = i\}$$

Transient analysis

A CTMC can be specified in terms of the occupancy probability vector π and a transition probability matrix P

$$\pi^{(t)} = \pi^{(0)} P^{(t)}$$

Transient analysis

State-transition-rate matrix, the Q matrix

$$q_{ij} = \begin{cases} \text{rate of going from state } i \text{ to state } j & i \neq j, \\ -\sum_{k \neq i} q_{ik} & i = j. \end{cases}$$

$$\sum_{j \in \mathcal{S}} q_{ij} = 0$$

We have that:

$$P^{(t)} = e^{Qt} \qquad \text{for } t>=0$$

$$e^{Qt} = I + \frac{tQ}{1!} + \frac{t^2Q^2}{2!} + \frac{t^3Q^3}{3!} + \cdots$$

This allows to compute the probability of reaching state j from state i at time t : $p^{(t)}_{ij}$

We have that:

$$\pi^{(t)} = \pi^{(0)} e^{Qt}$$

Different numerical solution methods

Sojourn time

For the memoryless property,
the sojourn time spent by a CTMC in any of its states is independent of how long the CTMC has previously been in state i.

There is only one random variable that has this property: the exponential random variable:

ST_i sojourn time in state i:

$$ST_i = e^{(a_i)}$$

-> the time spent in each state takes non-negative real values and has an exponential distribution

Steady-state behaviour

THEOREM

For **aperiodic irreducible continuous-time** Markov chain for each j,

$$\lim_{t \to \infty} \pi_j^{(t)}$$

exists and is independent from $\pi^{(0)}$

The solution can be computed under the constraint

$$\pi Q = 0 \qquad \textit{and} \qquad \sum_{i=1}^{n} \pi_i = 1$$

The steady-state distribution is independent of the initial-state distribution.

For general CTMC more complex solution methods are required

Direct methods:

Good packages exists

Very poor performance if Q is very large

Iterative methods:

An iterative method converges if :

$$\lim_{k \to \infty} \left\| \pi^{(k)} - \pi \right\| = 0$$

Stopping condition: $$\left\| \pi^{(k+1)} - \pi^{(k)} \right\| < \varepsilon$$

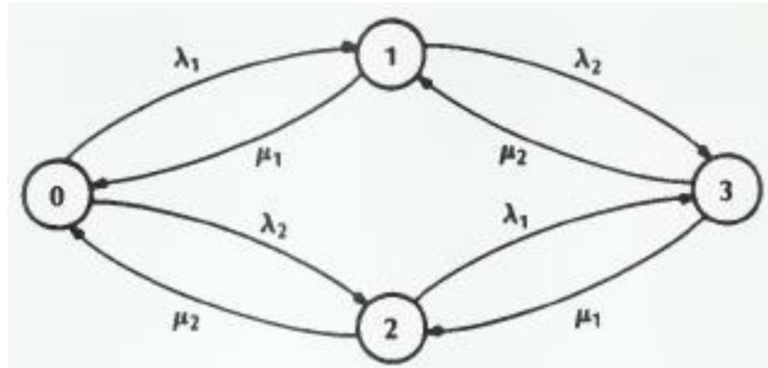
Other methods

…………………………

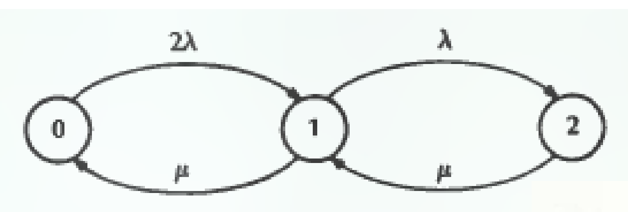
Dual processor system with repair

A, B processors

Rates: $\lambda 1$, $\lambda 2$ and $\mu 1$, $\mu 2$
Identification of states:
- A, B working
- A working, B failed
- B working, A failed
- A, B failed



Collapsed model
Single repair at a time



$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

$p(0) = [1, 0, 0]$

Availability

$$A(t) = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2} - \frac{4\lambda^2 \exp\left(-(1/2)[(3\lambda + 2\mu) + \sqrt{\lambda^2 + 4\lambda\mu}]t\right)}{\lambda^2 + 4\lambda\mu + (3\lambda + 2\mu)\sqrt{\lambda^2 + 4\lambda\mu}} - \frac{4\lambda^2 \exp\left(-(1/2)[(3\lambda + 2\mu) - \sqrt{\lambda^2 + 4\lambda\mu}]t\right)}{\lambda^2 + 4\lambda\mu - (3\lambda + 2\mu)\sqrt{\lambda^2 + 4\lambda\mu}}$$

$A(t) = 1 - p_2(t)$

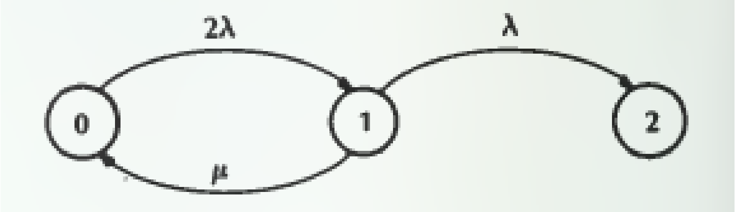
$$A_{ss} = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2}$$

Laplace transform

Reliability modeling

- making state 2 a trapping state

$p(0) = [1, 0, 0]$



$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda - \mu & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Reliability $R(t) = 1 - p_2(t)$ $R(t)=p_0(t) + p_1(t)$

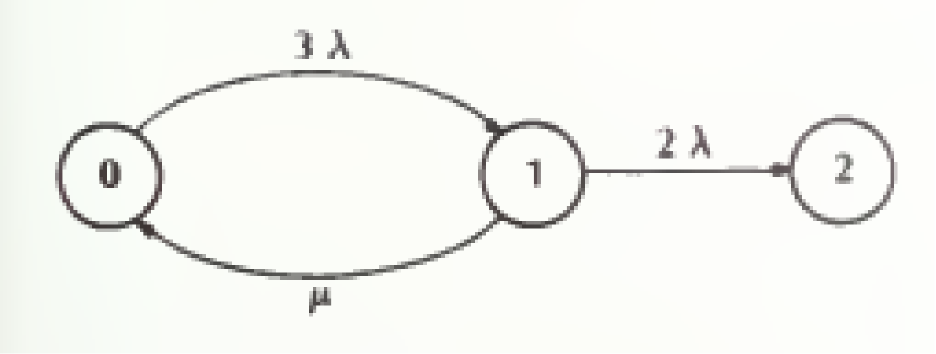
Laplace transform

$$R(t) = \frac{4\lambda^2 \exp(-(1/2)(3\lambda + \mu - \sqrt{\lambda^2 + 6\lambda\mu + \mu^2})t)}{(3\lambda + \mu)\sqrt{\lambda^2 + 6\lambda\mu + \mu^2} - \lambda^2 - 6\lambda\mu - \mu^2} - \frac{4\lambda^2 \exp(-(1/2)(3\lambda + \mu + \sqrt{\lambda^2 + 6\lambda\mu + \mu^2})t)}{(3\lambda + \mu)\sqrt{\lambda^2 + 6\lambda\mu + \mu^2} + \lambda^2 + 6\lambda\mu + \mu^2}$$

TMR system with repair

Rates: λ and μ

Identification of states:
- 3 processors working, 0 failed
- 2 processors working, 1 failed
- 1 processor working, 2 failed



Transition rate matrix:

$$Q = \begin{bmatrix} -3\lambda & 3\lambda & 0 \\ \mu & -2\lambda-\mu & 2\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

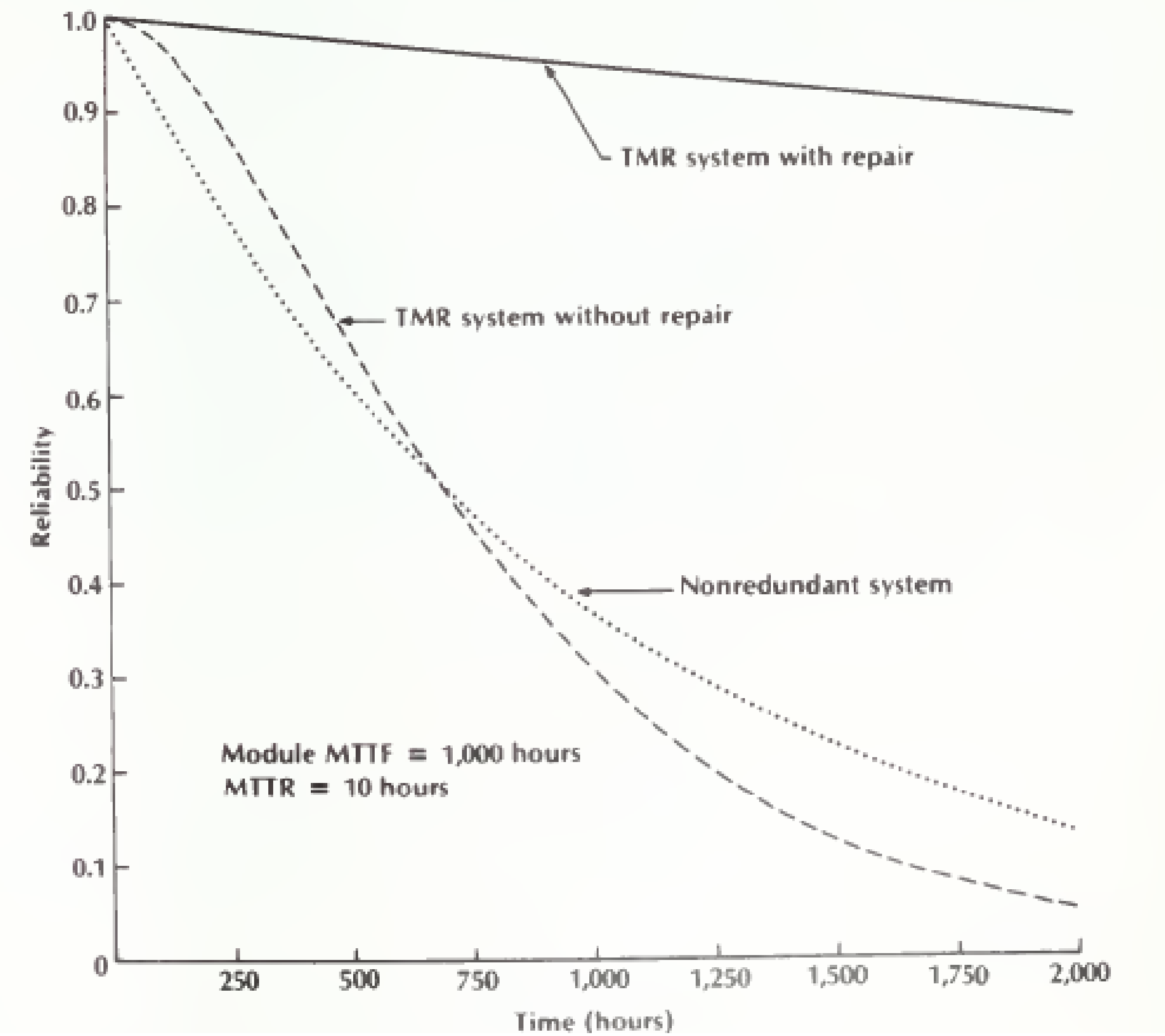
$P(0) = [1, 0, 0]$

Reliability $R(t) = 1 - p2(t)$

Laplace transform

$$R(t) = \frac{5\lambda + \mu + \sqrt{\lambda^2 + 10\lambda\mu + \mu^2}}{2\sqrt{\lambda^2 + 10\lambda\mu + \mu^2}} \exp\left(-(1/2)(5\lambda + \mu - \sqrt{\lambda^2 + 10\lambda\mu + \mu^2})t\right) - \frac{5\lambda + \mu - \sqrt{\lambda^2 + 10\lambda\mu + \mu^2}}{2\sqrt{\lambda^2 + 10\lambda\mu + \mu^2}} \exp\left(-(1/2)(5\lambda + \mu + \sqrt{\lambda^2 + 10\lambda\mu + \mu^2})t\right)$$

Comparison with nonredundant system and TMR without repair



MTTF

$$MTTF = \int_{t=0}^{\infty} R(t)\, dt$$

period the system is in a state that correspond to correct behavior

TMR with repair:

$$MTTF = \int_{t=0}^{\infty} p_0(t) + p_1(t)\ \ dt$$

failure rate $\lambda = 0.001$ repair rate $\mu = 0.1$

$$\text{TMR with repair MTTF} = \frac{5}{6\lambda} + \frac{\mu}{6\lambda^2} = 17{,}5000 \text{ hours}$$

MTTF is equal to the MTTF of a TMR system without repair plus an additional term due to the repair activity.

$$\text{Nonredundant MTTF} = \frac{1}{\lambda} = 1000 \text{ hours}$$

$$\text{TMR without repair MTTF} = \frac{5}{6\lambda} = 833 \text{ hours}$$

on-line repair allows the system MTTF to increase by a factor of 17

System model analysis

What is the availability of the system at time t?

What is the steady-state availability?

What is the expected time to failure?

………….

The Markov model fits with the standard assumption of failure rates a constant, leading to exponentially distributed inter-arrival times of failures. Similarly, we assume costant **repair rate**.

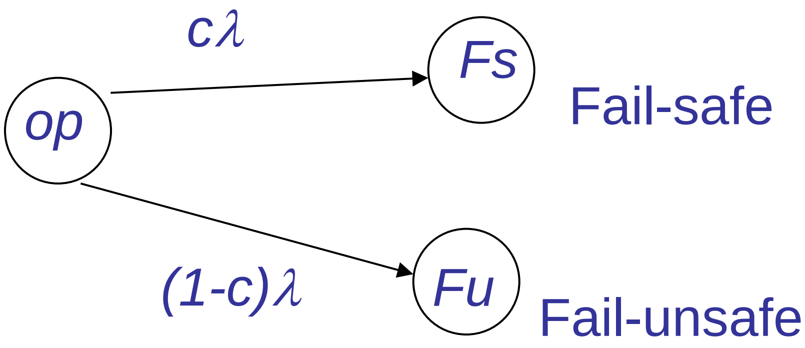
What about safety?

Safety

Safety - avoidance of catastrophic consequences -
As a function of time, S(t), is the probability that the system either behaves correctly or will discontinue its functions in a manner that causes no harm (operational or Fail-safe)

Coverage – The coverage is the measure **c** of the system ability to reach a fail-safe state after a fault.

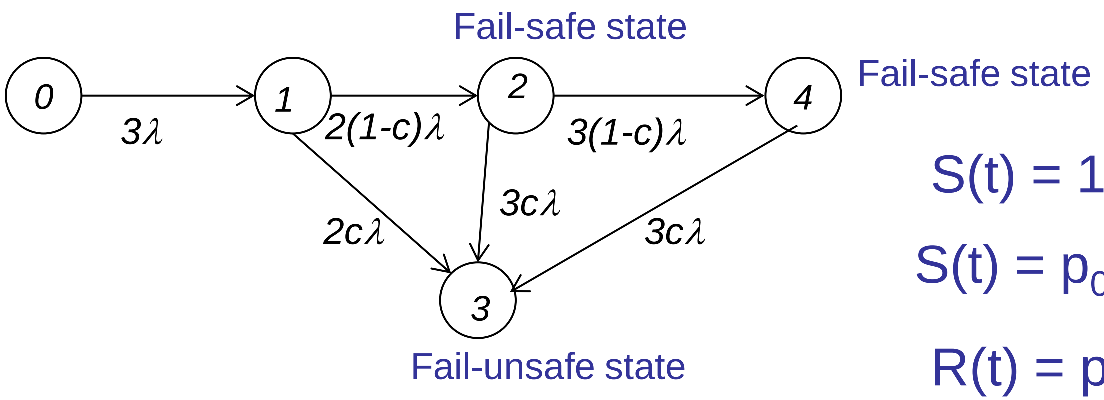
Modeling coverage and safety in a Markov chain means that every unfailed state has two transitions to two different states, one of which is fail-safe, the other is fail-unsafe.



TMR

the system can be in a safe state although the failures of two components, if the output of the three components disagree

c = probability of coincident failures of two components



$S(t) = 1 - p_3(t)$

$S(t) = p_0(t) + p_1(t) + p_2(t) + p_4(t)$

$R(t) = p_0(t) + p_1(t)$

0 three correct components
1 one faulty component
2 two faulty components (no coincident failures)
3 two faulty component coincident failures
4 three faulty components (no coincident failures)

Observations

Quantitative dependability evaluation:

- guiding design decisions
- assessing systems as built
- mandatory for safety critical systems

Model construction techniques

-> scalability challenge

- **composition approaches**

  build complex models in a modular way through a composition of its submodels

- **decomposition/aggregation approaches**

  (hierarchical decomposition approach)

  The overall model is decoupled in simpler and more tractable submodels, and the measures obtained from the solution of the sub-models are then aggregated to compute those concerning the overall model.