Discrete-time Markov chain (DTMC)
State space distribution

$\pi^{(t)} = [\pi_0^{(t)}, \pi_1^{(t)}, \pi_2^{(t)}, \ldots]$ state occupancy vector at time t

Probability that the Markov process is in state i at time-step t

$$\pi_i^{(t)} = P\{X_t = i\}$$

initial state space distribution: $\pi^{(0)} = (\pi_1^{(0)}, \ldots, \pi_n^{(0)})$

A single step forward: $\pi^{(1)} = \pi^{(0)} P$

Transient solution: $\pi^{(t)}$

State occupancy vector at time t in terms of the transition probability matrix:

$$\pi^{(t)} = \pi^{(0)} P^t$$

System evolution in a finite number of steps computed starting from the initial state distribution and the transition probability matrix

Limiting behaviour

A Markov process can be specified in terms of the state occupancy probability vector π and a transition probability matrix P

$$\pi^{(t)} = \pi^{(0)} P^t$$

The limiting behaviour of a DTMC (steady-state behaviour):

$$\lim_{t \to \infty} \pi_j^{(t)}$$

The limiting behaviour of a DTMC depends on the characteristics of its states. Sometimes the solution is simple.

Irreducible DTMC

A state j is said to be **accessible** from state i if there exists t >0 such that $P_{ij}^{(t)} > 0$, we write i->j

A DTMC is **irreducible** if each state is accessible from every other state in a finite number of steps :

for each i, j: i -> j

A subset S' of S is closed if there not exists any transition from S' to S-S'

Classification of states

A state i is **recurrent** if

(i->j) then (j->i)

process moves again to state i with probability 1

recurrent non-null: medium time of recurrence is finite

recurrent null: medium time of recurrence is infinite

A state i is **transient** if

exists (j!=i) such that (i->j) and not (j->i)

A state i is **absorbent** if

p_{ii}=1

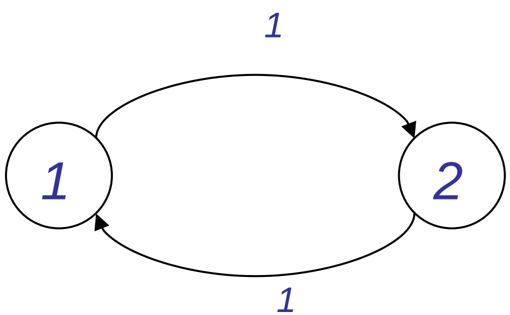
(i is a recurrent state)

Classification of states

Given a recurrent state, let d be the *greatest common divisor of all the* integers m such that $P_{ii}^{(m)} > 0$

A recurrent state i is **periodic** if d > 1

A recurrent state i is **aperiodic** if d = 1: it is possible to move to the same state in one step



state 1 is periodic with period d=2

state 2 is periodic with period d=2

Steady-state behaviour

THEOREM:

For **aperiodic irreducible** Markov chain for each j

$$\lim_{t \to \infty} \pi_j^{(t)}$$

exists and the solution is independent from $\pi^{(0)}$

Moreover, if **all states are recurrent non-null**, the **steady-state behaviour** of the Markov chain is given by the *fixpoint* of the equation:

$$\pi^{(t)} = \pi^{(t-1)} \, P$$

with

$$\Sigma_j \, \pi_j = 1$$

π_j is inversely proportional to the period of recurrence of state j

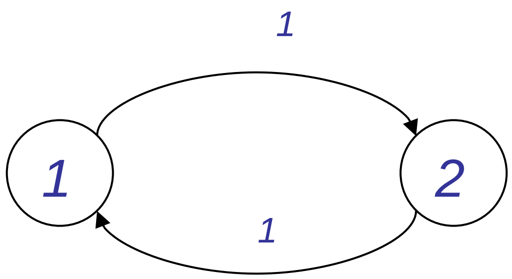
Time-average state space distribution

For periodic Markov chains $\lim_{t \to \infty} \pi_j^{(t)}$

doesn't exist (caused by the probability of the periodic state)

We compute the time-average state space distribution, called π^*

$$\pi^* = \lim_{t \to \infty} \frac{\sum_{i=1}^{t} \pi^{(i)}}{t}$$



$\pi^{(0)} = (1,0)$

state i is periodic with period d=2

$$P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$\pi^{(0)} = (1,0)$

$\pi^{(1)} = \pi^{(0)} P$ $\quad \pi^{(1)} = (0,1)$

$\pi^{(2)} = \pi^{(1)} P$ $\quad \pi^{(2)} = (1,0)$

....

Continuous-time Markov models

Continuous-time models:

state transitions occur at random intervals

transition rates assigned to each transition

Markov property assumption:

the length of time already spent in a state does not influence either the probability distribution of the next state or the probability distribution of remaining time in the same state before the next transition

These very strong assumptions imply that the waiting time spent in any one state is exponentially distributed

Thus the Markov model naturally fits with the standard assumptions that failure rates are constant, leading to exponential distribution of interarrivals of failures

Continuous-time Markov models

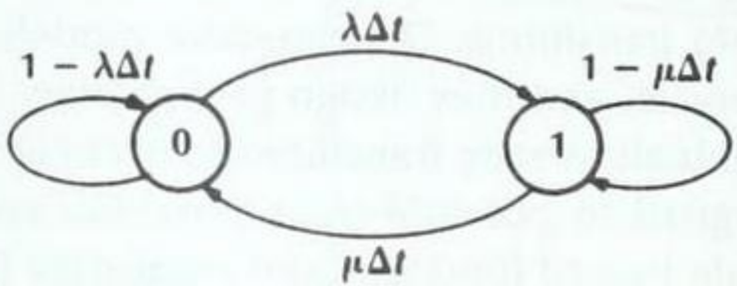
derived from the discrete time model, taking the limit as the time-step interval approaches zero

Single system with repair

λ failure rate, μ repair rate

state 0: working
state 1: failed

$p_0(t)$ probability of being in the operational state
$p_1(t)$ probability of being in the failed state



$\lambda\Delta t$, $\mu\Delta t$—State transition probabilities
λ, μ—State transition rates

$$P = \begin{bmatrix} 1 - \lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1 - \mu\Delta t \end{bmatrix}$$

Transition Matrix P

Continuous-time Markov models

Probability of being in state 0 or 1 at time t+Δt:

$$[p_0(t+\Delta t),\ p_1(t+\Delta t)] = [p_0(t),\ p_1(t)] \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

↑

probability of being in
state 0 at time t+Δt

Performing multiplication, rearranging and dividing by Δt, taking the limit as Δt approaches to 0:

$$\frac{dp_0(t)}{dt} = \dot{p}_0(t) = -\lambda p_0(t) + \mu p_1(t)$$

$$\frac{dp_1(t)}{dt} = \dot{p}_1(t) = \lambda p_0(t) - \mu p_1(t)$$

Continuous-time Chapman-Kolmogorov equations

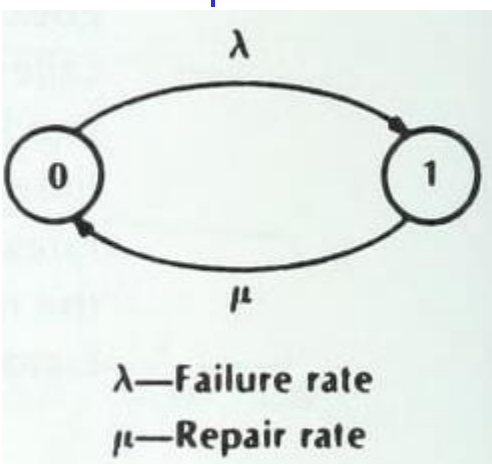
Continuous-time Markov models

Matrix form:

T matrix

$$[\dot{p}_0(t), \dot{p}_1(t)] = [p_0(t), p_1(t)] \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

The set of equations can be written by inspection of a transition diagram without self-loops and Δt's:



Continuous time Markov model graph

The change in state 0 is minus the flow out of state 0 times the probability of being in state 0 at time t, plus the flow into state 0 from state 1 times the probability of being in state 1.

Continuous-time Markov models

Chapman-Kolmogorov equations solved by use of a LaPlace transform of a time domain function

$$[p_0(0), p_1(0)] = [p_0^x(s), p_1^x(s)] \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}$$

↑

probability of being in state 0 at time t=0

A matrix

Linear equation solving techniques

$$\vec{P}(0) = \vec{P}^x(s)[s\mathbf{I} - T] = \vec{P}^x(s)A$$

$$\vec{P}^x(s) = \vec{P}(0)[s\mathbf{I} - T]^{-1} = \vec{P}(0)A^{-1}$$

where **I** is the identity matrix

We solve the equations. We obtain as solutions a ratio of two polynomials in s. Then we apply the inverse transform to the solutions.

Continuous-time Markov models

Our example

$$A^{-1} = \frac{\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}}{s^2 + \lambda s + \mu s}$$

Assume the system starts in the operational state: P(0) = [1,0]

$$\overline{P}^x(s) = [1,0] \begin{bmatrix} \dfrac{s+\mu}{s^2+\lambda s+\mu s} & \dfrac{\lambda}{s^2+\lambda s+\mu s} \\[2ex] \dfrac{\mu}{s^2+\lambda s+\mu s} & \dfrac{s+\lambda}{s^2+\lambda s+\mu s} \end{bmatrix}$$

$$p_0^x(s) = \frac{s+\mu}{s^2+\lambda s+\mu s}$$

$$p_1^x(s) = \frac{\lambda}{s^2+\lambda s+\mu s}$$

We apply the inverse transforms.

Continuous-time Markov models

$$p_0(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$p_1(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

⇦ A(t)

$p_0(t)$ probability that the system is in the operational state at time t, availability at time t

The availability consists of a steady-state term and an exponential decaying transient term

Only steady-state solution

Chapman-Kolmogorov equations: derivative replaced by 0; p0(t) replaced by p0(0) and p1(t) replaced by p1(0)

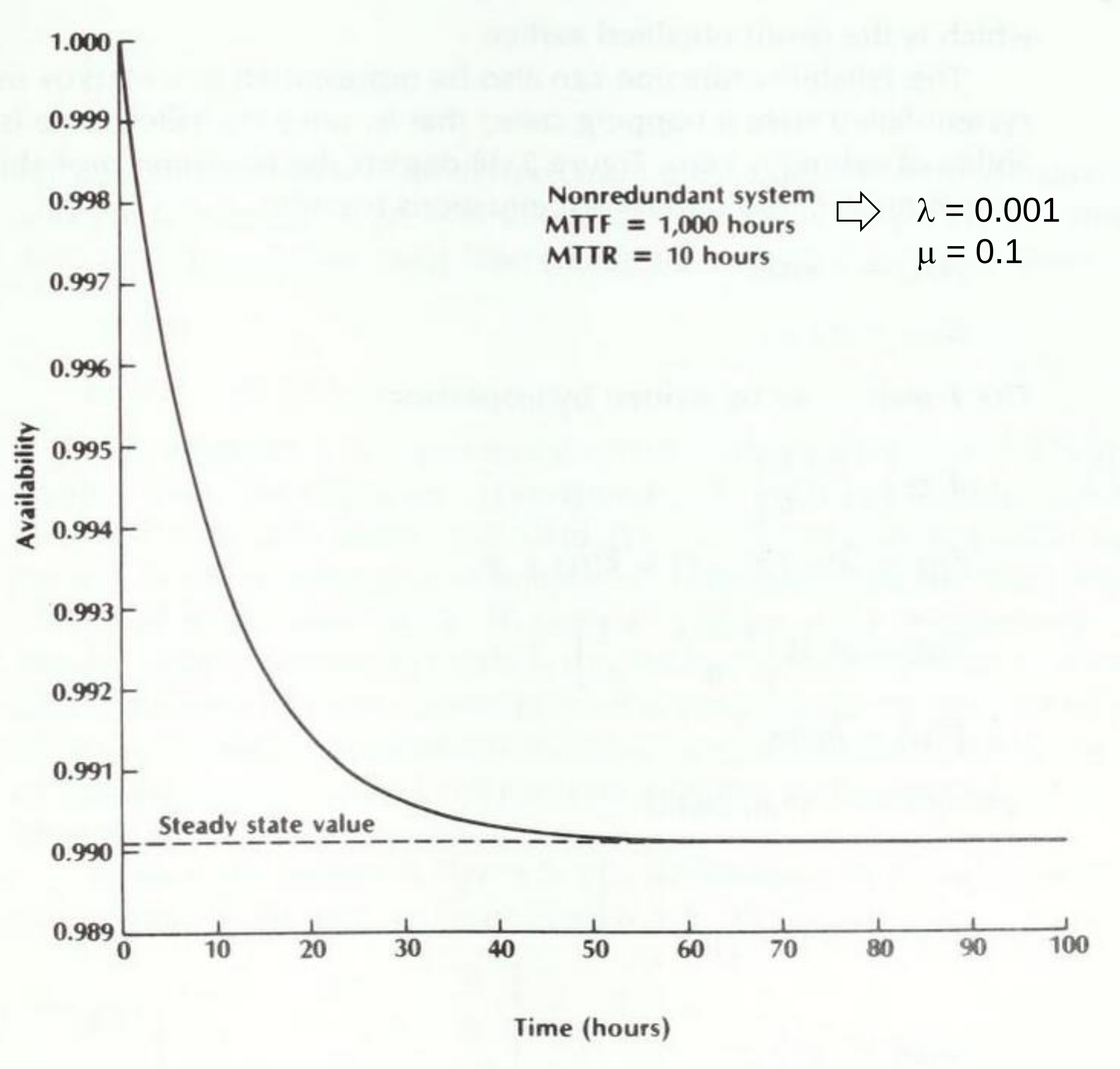
$$0 = -\lambda p_0 + \mu p_1$$

$$0 = \lambda p_0 - \mu p_1$$

⇨

$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu}} = \frac{\mu}{\lambda + \mu}$$

Availability as a function of time



$\lambda = 0.001$

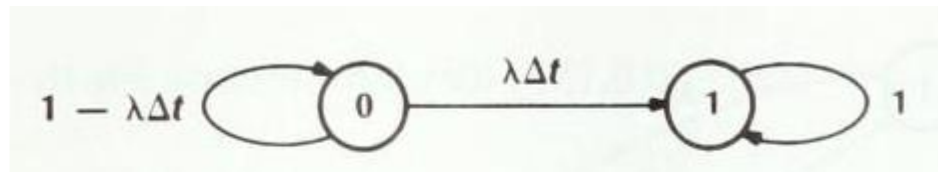
$\mu = 0.1$

The steady-state value is reached in a very short time

Continuous-time Markov models: Reliability

Markov model making the system-failed state a trapping state

Single system without repair



$\lambda\Delta t$ = state transition probability

Differential equations:

$$\dot{p}_0(t) = -\lambda p_0(t)$$

$$\dot{p}_1(t) = \lambda p_0(t)$$

T matrix

$$\begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$

Continuous time Markov model graph

λ = failure rate

T matrix can be built by inspection

Continuous-time Markov models: Reliability

A matrix
A= [sI –T]

$$\begin{bmatrix} s+\lambda & -\lambda \\ 0 & s \end{bmatrix}$$

$$\vec{P}^x(s) = \vec{P}(0)A^{-1}$$

$$\vec{P}^x(s) = [1,0]\,\frac{\begin{bmatrix} s & \lambda \\ 0 & s+\lambda \end{bmatrix}}{s^2+\lambda s}$$

$$p_0^x(s) = \frac{1}{s+\lambda}$$

$$p_1^x(s) = \frac{1}{s} - \frac{1}{s+\lambda}$$

Taking the inverse transform:

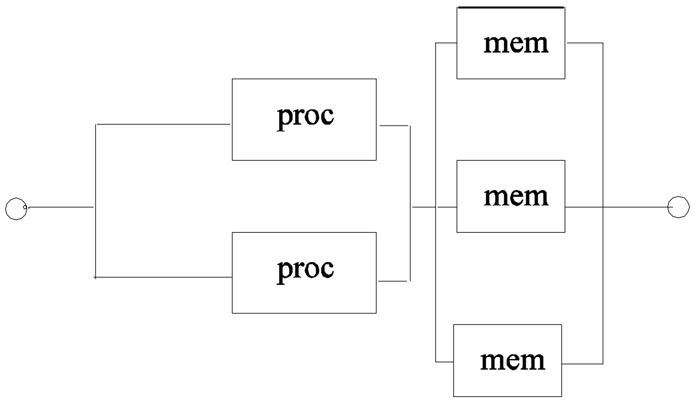
$$p_0(t) = e^{-\lambda t}$$

$$p_1(t) = 1 - e^{-\lambda t}$$

Continuous-time homogeneous Markov chains (CTMC)

An example of modeling (CTMC)

Multiprocessor system with 2 processors and 3 shared memories system. System is operational if at least one processor and one memory are operational.



λ_m failure rate for memory

λ_p failure rate for processor

X random process that represents the number of operational memories and the number of operational processors at time t

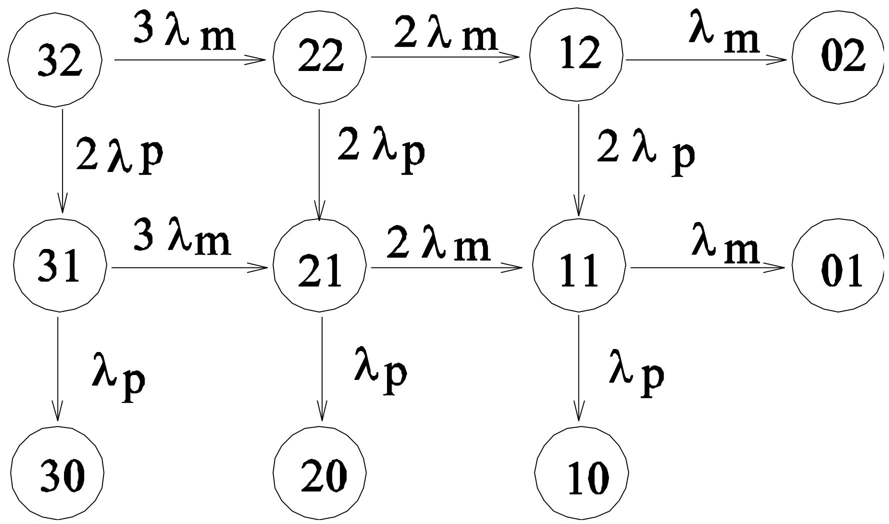
Given a state (i, j):

i is the number of operational memories;

j is the number of operational processors

S = {(3,2), (3,1), (3,0), (2,2), (2,1), (2,0), (1,2), (1,1), (1,0), (0,2), (0,1)}

Reliability modeling



λ_m failure rate for memory

λ_p failure rate for processor

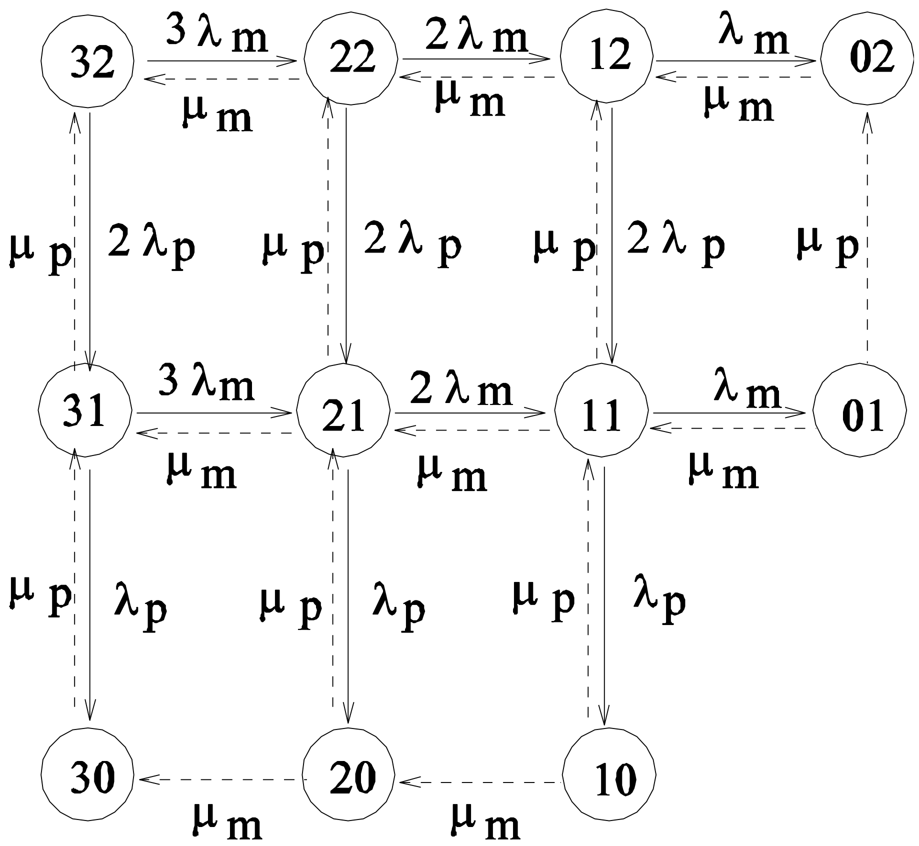
(3, 2) -> (2,2) failure of one memory

(3,0), (2,0), (1,0), (0,2), (0,1) are absorbent states

Availability modeling

- Assume that faulty components are replaced and we evaluate the probability that the system is operational at time t
- Constant repair rate μ (number of expected repairs in a unit of time)
- Strategy of repair:
  only one processor or one memory at a time can be substituted
- The behaviour of components (with respect of being operational or failed) is not independent: it depends on whether or not other components are in a failure state.

- Strategy of repair:
  only one component can be substituted at a time



λm failure rate for memory

λp failure rate for processor

μm repair rate for memory

μp repair rate for processor

- An alternative strategy of repair:

  only one component can be substituted at a time and processors have higher priority
- exclude the lines μm representing memory repair in the case where there has been a process failure