DTMC: Discrete-time homogeneous Markov chain

- Markov process X (memoryless property) with finite *discrete-state space S*
- steady-state transition probabilities
- transitions at fixed intervals (steps) $t \in \{0, 1, 2, \ldots\}$

Definitions

n number of states

$\pi = (\pi_1, \ldots, \pi_n)$ state space distribution

Transition probability matrix

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \\ p_{n1} & \cdots & p_{nn} \end{bmatrix},$$

$$p_{ij} = \mathcal{P}\{X_1 = j | X_0 = i\}$$

probability of moving from state i to state j in one step

Memoryless property

$$\mathcal{P}\{X_{t+1} = j | X_0 = k_0, \ldots, X_{t-1} = k_{t-1}, X_t = i\} = \mathcal{P}\{X_{t+1} = j | X_t = i\}$$

Characterization of time evolution of the process

$$p_{ij}^{(t)} = \mathcal{P}\{X_t = j | X_0 = i\}, \; t \in \{0, 1, 2, ...\}$$

probability of moving from state i to state j in t steps

Transition probability from an initial state to a final state in (t1+t2) steps:

- transition probability from the initial state to a state k in t1 steps
- transition probability from state k to final state in t2 steps

$$p_{ij}^{(t1+t2)} = \sum_{k \in \mathcal{S}} p_{ik}^{(t1)} p_{kj}^{(t2)} \; \forall i, j \in \mathcal{S} \; \forall t1, t2 \geq 1$$

It follows that: t-steps Transition probability matrix

$$P^{(t)} = P^t$$ *t-th power of P*

State space distribution

Initial state space distribution: $\pi^{(0)} = (\pi_1^{(0)}, \ldots, \pi_n^{(0)})$

$\pi_i^{(0)} = P\{X_0 = i\}$ initial probability vector

A single step forward:

$$\pi^{(1)} = \pi^{(0)} P$$

Probability that the DTMC is in state i at time-step t

$$\pi_i^{(t)} = P\{X_t = i\}$$

Transient solution: $\pi^{(t)}$

State occupancy vector at time t in terms of the transition probability matrix:

$$\pi^{(t)} = \pi^{(0)} P^t$$

System evolution in a finite number of steps computed starting from the initial state distribution and the transition probability matrix

Sojourn time

time spent by a DTMC in any of its states (independently of its initial distribution)

$ST_i(k)$ probability that the DTMC stays in state i for k steps before moving to another state

Geometric distribution – random variable with memoryless property

Z_i = number of steps that the DTMC stays in state i before moving

S= {0, 1} 0 stay in state i 1 move into a state different from i

$$P\{Z_i = k\} = p_{ii}^{(k-1)} (1 - p_{ii})$$

p_{ii} probability of staying in state i

$(1-p_{ii})$ probability of moving to another state

Z = number of trials before the first success (included the success)

$$ST_i(k) = P\{Z_i = k\}$$

Limiting behaviour

A DTMC can be specified in terms of the state occupancy probability vector π and a transition probability matrix P

$$\pi^{(t)} = \pi^{(0)} P^t$$

The limiting behaviour of a DTMC (steady-state behaviour):

$$\lim_{t \to \infty} \pi_j^{(t)}$$

The limiting behaviour of a DTMC depends on the characteristics of its states. Sometimes the solution is simple.

Irreducible DTMC

A state j is said to be **accessible** from state i if there exists t >=0 such that $P_{ij}^{(t)} > 0$, we write i->j

In terms of the graph, j is accessible from state i if there is a path from node i to node j.

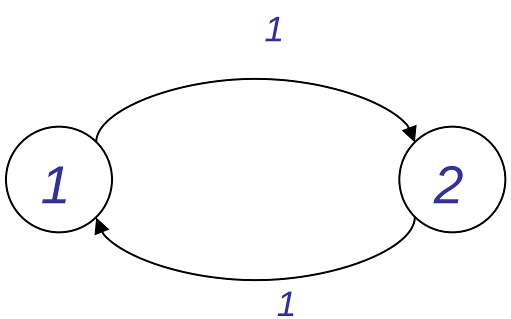
A DTMC is **irreducible** if each state is accessible from every other state (for each i, j: it holds i ->j), otherwise it is **reducible**

Classification of states

A state i is **periodic** with period d >1 if it is possible to move to state i only after n steps such that n = d, 2d, 3d, ….:

$$P_{ji}^{(t)} > 0 \quad \text{implies t is an integer multiple of d}$$

If d=1, the state i is said to be **aperiodic;** it is possible to move to the same state in one step



state 1 is periodic with period d=2

state 2 is periodic with period d=2

Classification of states

A state i is **recurrent** if
for each j: if (i->j) then (j->i)

A state i is **transient** if
exists (j!=i) such that (i->j) and not (j->i)

A state i is **absorbent** if
$p_{ii}=1$
(i is a recurrent state)

Each Markov chain has at least one recurrent state

Steady-state behaviour

For **aperiodic irreducible** Markov chain for each j

$$\lim_{t \to \infty} \pi_j^{(t)}$$

exists and are independent from $\pi^{(0)}$

If **all states are recurrent**, the **steady-state behaviour** of the Markov chain is given by the *fixpoint* of the equation:

$$\pi^{(t)} = \pi^{(t-1)} P$$

with

$$\Sigma_j \pi_j = 1$$

π_j is inversely proportional to the period of recurrence of state j

Time-average state space distribution

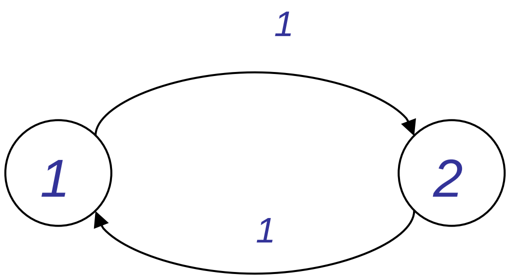
For periodic Markov chains $\lim_{t\to\infty} \pi_j^{(t)}$

doesn't exist (caused by the probability of the periodic state)

We compute the time-average state space distribution, called π^*

$$\pi^* = \lim_{t\to\infty} \frac{\sum_{i=1}^{t} \pi^{(i)}}{t}$$

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$



$\pi^{(0)} = (1,0)$

state i is periodic with period d=2

$\pi^{(0)} = (1,0)$

$\pi^{(1)} = \pi^{(0)} P \qquad \pi^{(1)} = (0,1)$

$\pi^{(2)} = \pi^{(1)} P \qquad \pi^{(2)} = (1,0)$

....

Continuous-time homogeneous Markov chain (CTMC)

- Markov process X (memoryless property) with discrete-state space S
- steady-state transition rates
- events may occur at any point in time

Definitions

Let T be an interval of real numbers (e.g., T=[0,1]).
Memoryless property:

$$\mathcal{P}\{X_{t+\tau}=j|X_t=i, X_{t-t_1}=k_1, ..., X_{t-t_n}=k_n\}=\mathcal{P}\{X_{t+\tau}=j|X_t=i\}$$

for all τ>0 and 0<t1<t2<…<tn.

Steady-state transition probabilities

$$\mathcal{P}\{X_{t+\tau}=j|X_t=i\}=\mathcal{P}\{X_\tau=j|X_0=i\}$$

State-transition-rate matrix, the Q matrix

$$q_{ij} = \begin{cases} \text{rate of going from state } i \text{ to state } j & i \neq j, \\ -\sum_{k \neq i} q_{ik} & i = j. \end{cases}$$

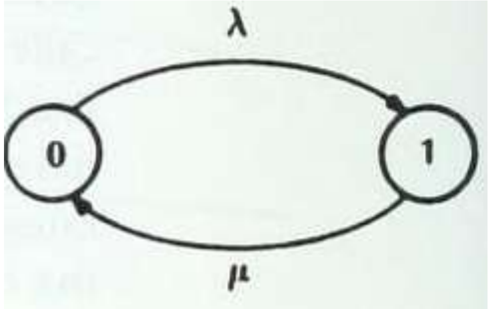
$$\sum_{j \in \mathcal{S}} q_{ij} = 0$$

Single system with repair

- transition rates: λ failure rate, μ repair rate
- identification of states
- initial state-space p(0) = [1, 0]

$\pi^{(t)} = [\pi_0^{(t)}, \pi_1^{(t)}]$

p(t) = [p_0(t), p_1(t)] in the book



$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - \mu p_1(t)$$

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Solution of the differential equations:

$$p_0(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$p_1(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

Availability

A(t) = p_0(t)

Transient solution

A CTMC can be specified in terms of the occupancy probability vector π and a transition probability matrix P

$$\pi^{(t)} = \pi^{(0)} P^t$$

where

$$P^{(t)} = e^{Qt} \qquad \text{for } t>=0$$

$$P^{(t)} = I + \frac{tQ}{1!} + \frac{t^2Q^2}{2!} + \frac{t^3Q^3}{3!} + \cdots$$

This allows to compute the probability of reaching state j from state i at time t : $p^{(t)}_{ij}$

We have:

$$\pi^{(t)} = \pi^{(0)} e^{Qt}$$

$$\frac{d}{dt}\pi^{(t)} = \pi^{(t)} Q$$

Different numerical solution methods

Sojourn time

For the memoryless property,
the sojourn time spent by a CTMC in any of its states is independent of how long the CTMC has previously been in state i.

There is only one random variable that has this property: the exponential random variable:

ST_i sojourn time in state i:

$$ST_i = e^{(a_i)}$$

-> the time spent in each state takes non-negative real values and has an exponential distribution

Steady-state behaviour

$$\lim_{t\to\infty} \pi_j^{(t)}, \forall j \in \mathcal{S}$$

We can prove that we have to solve the equation:

$\pi^*Q=0$, where $\pi^* = \lim_{t\to\infty} \pi_j^{(t)}$

For **irreducible CTMC** (irreducible if every state is accessible from every other state. For each i, j, it holds i ->j.) the solution can be calculated under the constraint:

$$\sum_{i=1}^{n} \pi_i^* = 1$$

The steady-state distribution is independent of the initial-state distribution

If the CTMC is not **irreducible** then more complex solution methods are required

Direct methods:

Good packages exists

Very poor performance if Q is very large

Iterative methods:

An iterative method converges if :

$$\lim_{k\to\infty}\left\|\pi^{(k)}-\pi^{*}\right\|=0$$

Stopping condition: $\left\|\pi^{(k+1)}-\pi^{(k)}\right\|<\varepsilon$

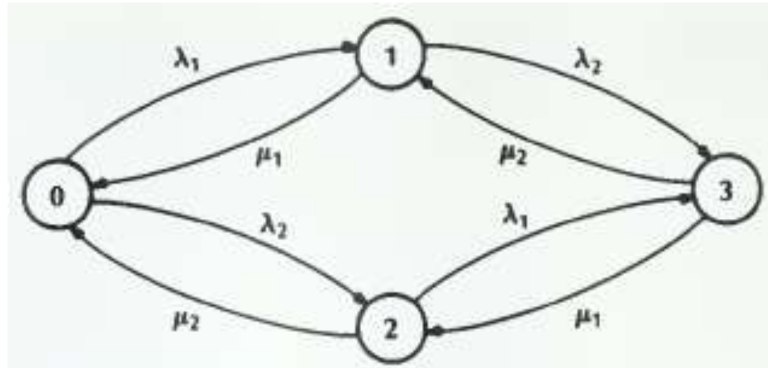
Other methods

……………………………

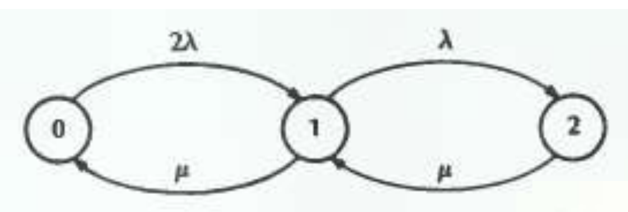
Dual processor system with repair

A, B processors

Rates: $\lambda 1$, $\lambda 2$ and $\mu 1$, $\mu 2$
Identification of states:
- A, B working
- A working, B failed
- B working, A failed
- A, B failed



Collapsed model
Single repair at a time



$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

p(0) = [1, 0, 0]

Availability

$$A(t) = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2} - \frac{4\lambda^2 \exp\left(-(1/2)[(3\lambda + 2\mu) + \sqrt{\lambda^2 + 4\lambda\mu}]t\right)}{\lambda^2 + 4\lambda\mu + (3\lambda + 2\mu)\sqrt{\lambda^2 + 4\lambda\mu}} - \frac{4\lambda^2 \exp\left(-(1/2)[(3\lambda + 2\mu) - \sqrt{\lambda^2 + 4\lambda\mu}]t\right)}{\lambda^2 + 4\lambda\mu - (3\lambda + 2\mu)\sqrt{\lambda^2 + 4\lambda\mu}}$$

$A(t) = 1 - p_2(t)$

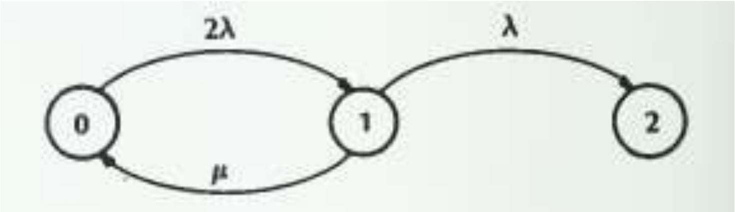
Laplace transform

$$A_{ss} = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2}$$

Reliability modeling

- making state 2 a trapping state

$p(0) = [1, 0, 0]$



$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda - \mu & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Reliability $R(t) = 1 - p_2(t)$ $\qquad R(t) = p_0(t) + p_1(t)$

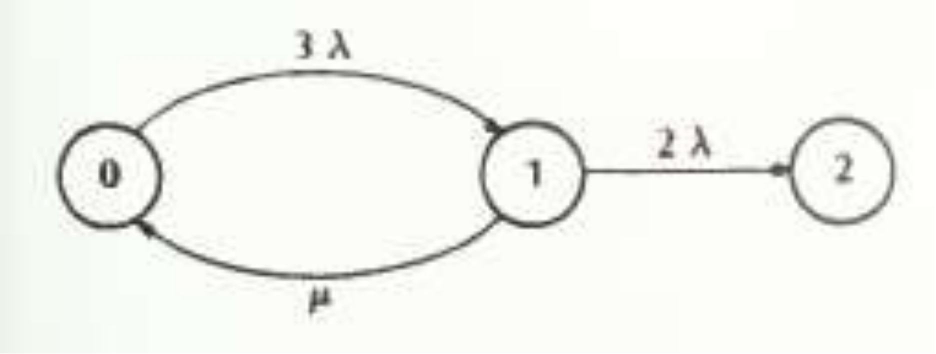
Laplace transform

$$R(t) = \frac{4\lambda^2 \exp\left(-(1/2)(3\lambda + \mu - \sqrt{\lambda^2 + 6\lambda\mu + \mu^2})t\right)}{(3\lambda + \mu)\sqrt{\lambda^2 + 6\lambda\mu + \mu^2} - \lambda^2 - 6\lambda\mu - \mu^2} - \frac{4\lambda^2 \exp\left(-(1/2)(3\lambda + \mu + \sqrt{\lambda^2 + 6\lambda\mu + \mu^2})t\right)}{(3\lambda + \mu)\sqrt{\lambda^2 + 6\lambda\mu + \mu^2} + \lambda^2 + 6\lambda\mu + \mu^2}$$

TMR system with repair

Rates: λ and μ

Identification of states:
- 3 processors working, 0 failed
- 2 processors working, 1 failed
- 1 processor working, 2 failed



Transition rate matrix:

$$Q = \begin{bmatrix} -3\lambda & 3\lambda & 0 \\ \mu & -2\lambda-\mu & 2\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

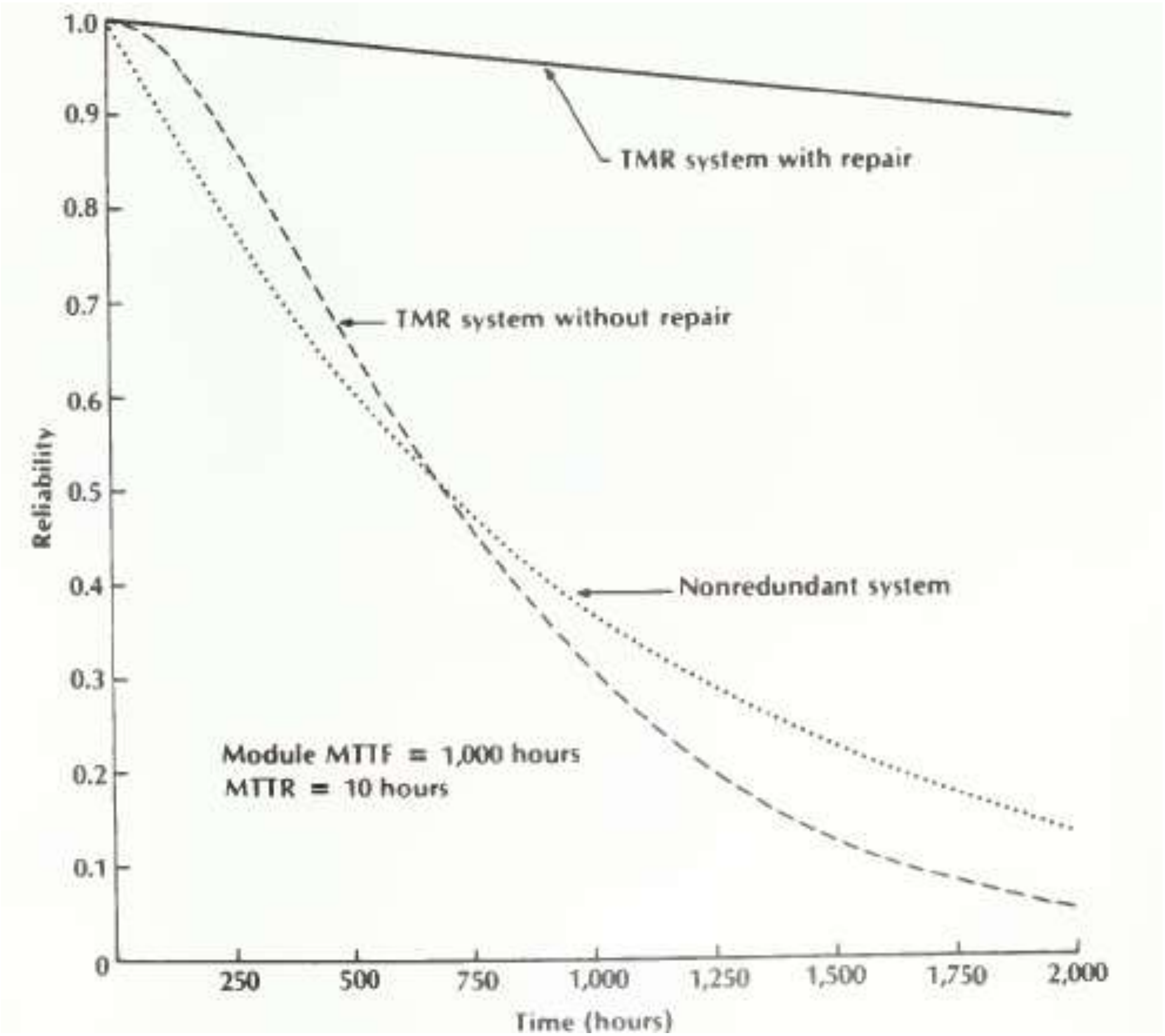
$P(0) = [1, 0, 0]$

Reliability $R(t) = 1 - p2(t)$

Laplace transform

$$R(t) = \frac{5\lambda + \mu + \sqrt{\lambda^2 + 10\lambda\mu + \mu^2}}{2\sqrt{\lambda^2 + 10\lambda\mu + \mu^2}} \exp\left(-(1/2)(5\lambda + \mu - \sqrt{\lambda^2 + 10\lambda\mu + \mu^2})t\right)$$
$$- \frac{5\lambda + \mu - \sqrt{\lambda^2 + 10\lambda\mu + \mu^2}}{2\sqrt{\lambda^2 + 10\lambda\mu + \mu^2}} \exp\left(-(1/2)(5\lambda + \mu + \sqrt{\lambda^2 + 10\lambda\mu + \mu^2})t\right)$$

Comparison with nonredundant system and TMR without repair



MTTF

$$\text{MTTF} = \int_{t=0}^{\infty} R(t)\, dt$$

period the system is in a state that correspond to correct behavior

TMR with repair:

$$\text{MTTF} = \int_{t=0}^{\infty} p_0(t) + p_1(t)\;\; dt$$

failure rate $\lambda = 0.001$ repair rate $\mu = 0.1$

$$\text{TMR with repair MTTF} = \frac{5}{6\lambda} + \frac{\mu}{6\lambda^2} = 17{,}5000 \text{ hours}$$

MTTF is equal to the MTTF of a TMR system without repair plus an additional term due to the repair activity.

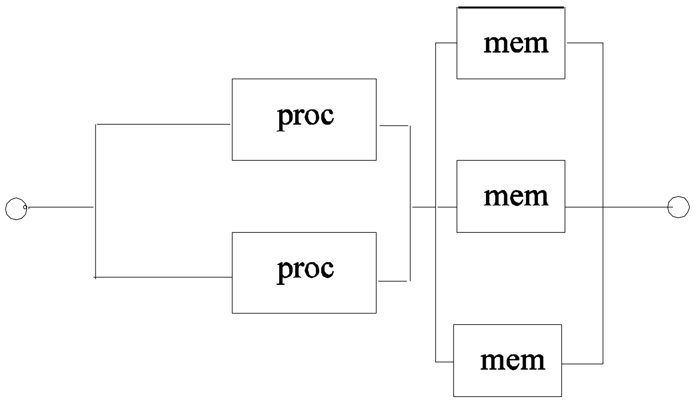
$$\text{Nonredundant MTTF} = \frac{1}{\lambda} = 1000 \text{ hours}$$

$$\text{TMR without repair MTTF} = \frac{5}{6\lambda} = 833 \text{ hours}$$

on-line repair allows the system MTTF to increase by a factor of 17

An example of modeling (CTMC)

Multiprocessor system with 2 processors and 3 shared memories system. System is operational if at least one processor and one memory are operational.



λ_m failure rate for memory

λ_p failure rate for processor

X random process that represents the number of operational memories and the number of operational processors at time t

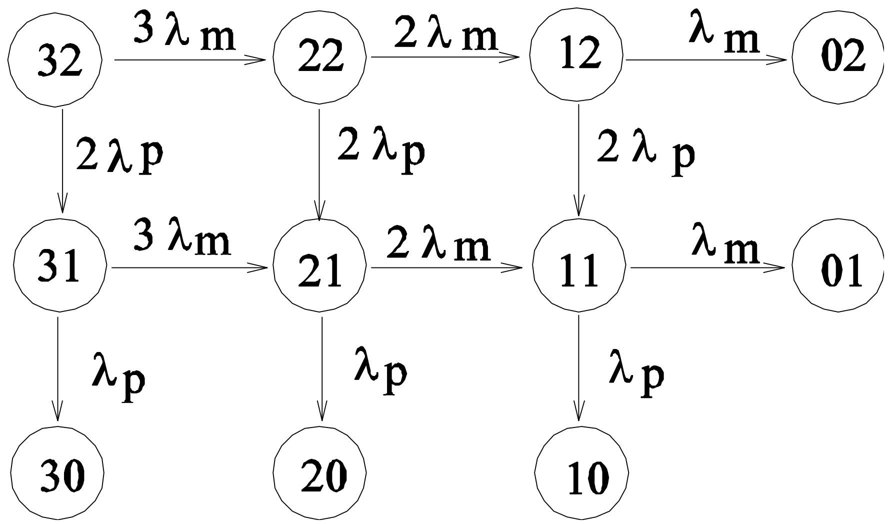
Given a state (i, j):

i is the number of operational memories;

j is the number of operational processors

S = {(3,2), (3,1), (3,0), (2,2), (2,1), (2,0), (1,2), (1,1), (1,0), (0,2), (0,1)}

Reliability modeling



λ_m failure rate for memory
λ_p failure rate for processor

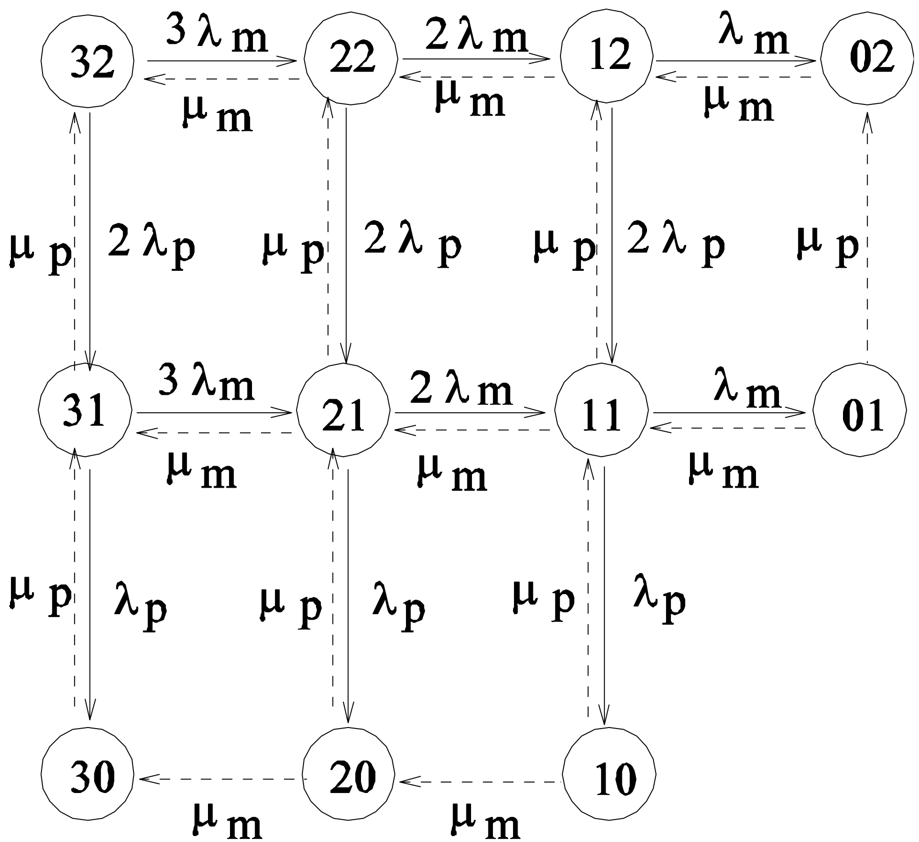
(3, 2) -> (2,2) failure of one memory

(3,0), (2,0), (1,0), (0,2), (0,1) are absorbent states

Availability modeling

- Assume that faulty components are replaced and we evaluate the probability that the system is operational at time t
- Constant repair rate μ (number of expected repairs in a unit of time)
- Strategy of repair:
  only one processor or one memory at a time can be substituted
- The behaviour of components (with respect of being operational or failed) is not independent: it depends on whether or not other components are in a failure state.

- Strategy of repair:
  only one component can be substituted at a time



λm failure rate for memory

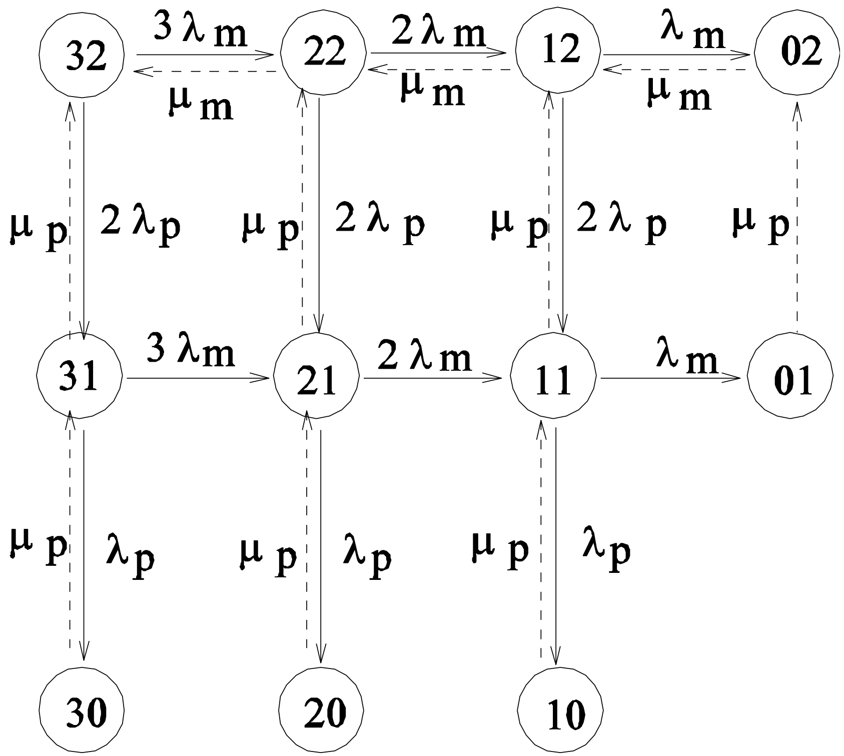
λp failure rate for processor

μm repair rate for memory

μp repair rate for processor

- An alternative strategy of repair:

  only one component can be substituted at a time and processors have higher priority
- exclude the lines μm representing memory repair in the case where there has been a process failure



System model analysis

What is the availability of the system at time t?

What is the steady-state availability?

What is the expected time to failure?

………….

The Markov model fits with the standard assumption of failure rates a constant, leading to exponentially distributed inter-arrival times of failures. Similarly, we assume costant **repair rate**.

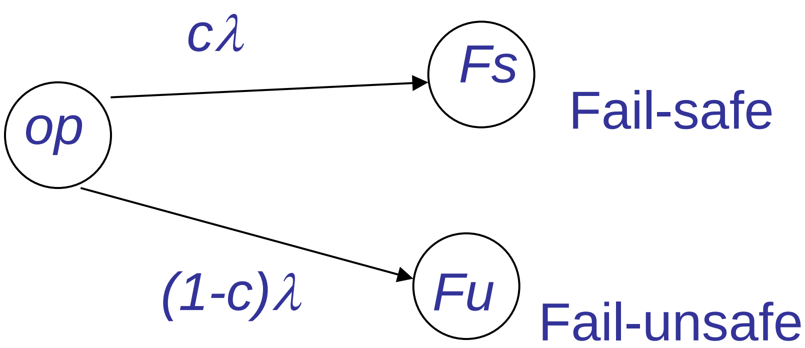
What about safety?

Safety

Safety - avoidance of catastrophic consequences - As a function of time, S(t), is the probability that the system either behaves correctly or will discontinue its functions in a manner that causes no harm (operational or Fail-safe)

Coverage – The coverage is the measure **c** of the system ability to reach a fail-safe state after a fault.

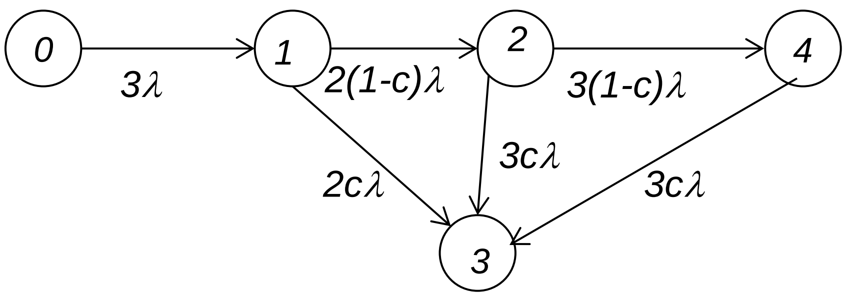
Modeling coverage and safety in a Markov chain means that every unfailed state has two transitions to two different states, one of which is fail-safe, the other is fail-unsafe.



TMR

the system can be in a safe state although the failures of two components, if the output of the three components disagree

c = probability of coincident failures of two components



$S(t) = 1 - p_3(t)$

$S(t) = p_0(t) + p_1(t) + p_2(t) + p_4(t)$

$R(t) = p_0(t) + p_1(t)$

0 three correct components
1 one faulty component
2 two faulty components (no coincident failures)
3 two faulty component coincident failures
4 three faulty components (no coincident failures)

Observations

Quantitative dependability evaluation:

- guiding design decisions
- assessing systems as built
- mandatory for safety critical systems

Model construction techniques

-> scalability challenge

- **composition approaches**

  build complex models in a modular way through a composition of its submodels

- **decomposition/aggregation approaches**

  (hierarchical decomposition approach)

  The overall model is decoupled in simpler and more tractable submodels, and the measures obtained from the solution of the sub-models are then aggregated to compute those concerning the overall model.