Availability

Availability - A(t)

the probability that the system is operating correctly and is available to perform its functions at the *instant* of time t

More general concept than reliability: **failure and repair** of the system

Repair rate - which is the average number of repairs that occur per time period, generally number of repairs per hours. Analogous to failure rate, constant repair rate

$$\mu(t) = \mu$$

Maintenability - $M(t)$ is the conditional probability that the system is repaired throughout the *interval of time* [0, t], given that the system was faulty at time 0

$$M(t) = 1 - e^{-\mu t}$$

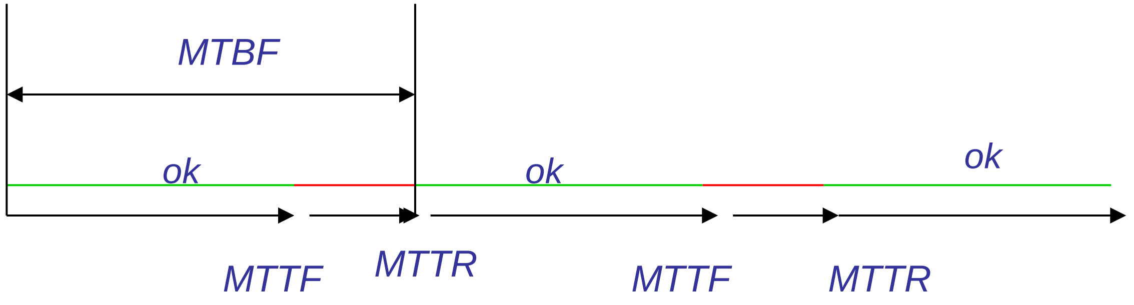
with μ constant repair rate.

MTTR - The Mean Time To Repair is the average time required to repair the system. Analogous to MTTF, MTTR is expressed in terms of the repair rate:

$$MTTR = \frac{1}{\mu}$$

Mean Time Between Failures - The MTBF is the average time between failures of the system, including the time required to repair the system and place it back into an operational status

MTBF = MTTF + MTTR



Steady-state availability

Steady-state availability (or instantaneous avaibaility) of a system is the percentage of time for which the system will deliver a correct service in presence of failures and repairs

Steady-state availability** (A_{ss})**:

$$A_{ss} = \frac{MTTF}{MTTF+MTTR}$$

Log-term probability that the system is available when requested
t -> ∞

Single system with failure rate λ and repair rate μ: $A_{ss} = \dfrac{\mu}{\lambda + \mu}$

Availability without maintenance and repair

$$MTTR = \infty \qquad A_{ss} = 0$$

Model-based evaluation of dependability

State-based models: Markov models

Model construction based on the identification of system states and changes of states

- each state represents a distinct combination of failed and working modules

- state transitions govern the changes of state that occur within a system

The system goes from state to state as modules fail and repair.

The state transitions are characterized by the probability of failure and the probability of repair

Model-based evaluation of dependability

Basic assumption underlying Markov models:

the system behavior at any time instant depends
only on the current state (independent of past values)

- systems with arbitrary structures and complex dependencies can be modeled

- assumption of independent failures no longer necessary

- used for both reliability and availability modeling

- based on a Markov process, a special type of random process

Random process

Random process

a collection of random variables $\{X_t\}$ *indexed by time*

Let X be a random variable representing the result of tossing a die

The sequence of results of tossing a die can be expressed by a random process

$$\{X_t\} \text{ with } t = 0, 1, 2, 3. \dots$$

$P[X_0 = 4] = 1/6$

$P[X_4 = 4 \mid X_3 = 2] = P[X_4 = 4] = 1/6$

In this case, random variables are independent

$X_0 = i \qquad X_1 = j \qquad X_2 = k \qquad \dots\dots\dots\dots\dots..$

The probability assigned to each transition is 1/6

Random process

Discrete-time random process
all state transitions occur at fixed intervals
probabilities assigned to each transition

Continuous-time random process
state transitions occur at random intervals
transition rates assigned to each transition

State space S of a random process $\{X_t\}$: the set of all possible values the process can take

$$S = \{y: X_t = y, \text{ for some } t\}$$

- Discrete-state random process
if the state space of random process is finite or countable (e.g., S={1, 2, 3,…})

- ***Continuous-state random process***
if the state space of random process is infinite and uncountable (e.g., S = the set of real numbers)

Discrete-time Markov process

Let $\{X_t, t>=0\}$ be a random process. A special type of random process is called the Markov process.

Basic assumption underlying Markov process:

the probability of state transition depends only on the current state

For each t, for any couple of states i and j, for any sequence $k_0, \dots, k_{t-1}$

$$\mathcal{P}\{X_{t+1} = j | X_0 = k_0, ..., X_{t-1} = k_{t-1}, X_t = i\} = \mathcal{P}\{X_{t+1} = j | X_t = i\}$$

the future behaviour is independent of past values (memoryless property)

Markov process: steady-state transition probabilities

Let $\{X_t, t>=0\}$ be a Markov process. The Markov process X has *steady-state transition probabilities* if for any pair of states i, j:

$$\mathcal{P}\{X_{t+1} = j | X_t = i\} = \mathcal{P}\{X_1 = j | X_0 = i\} \ \forall t \geq 0$$

The probability of transition from state i to state j does not depend by the time. This probability is called p_{ij}

$$p_{ij} = \mathcal{P}\{X_1 = j | X_0 = i\}$$

Transition probability matrix

If a Markov process is finite-state, we can define the transition probability matrix P (nxn)

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \\ p_{n1} & \cdots & p_{nn} \end{bmatrix},$$

$$p_{ij} = \mathcal{P}\{X_1 = j | X_0 = i\}$$

pij = probability of moving from state i to state j in one step

row i of matrix P:
probability of make a transition starting from state i

column j of matrix P:
probability of making a transition from any state to state j

Transition probability after n-time steps

THEOREM: Generalization of the steady-state transition probabilities.
For any i, j in S, and for any n>0

$$\mathcal{P}\{X_{t+n} = j | X_t = i\} = \mathcal{P}\{X_n = j | X_0 = i\} \ \forall t \geq 0$$

Definition: steady-state transition probability after n-time steps

$$p_{ij}^{(n)} = \mathcal{P}\{X_n = j | X_0 = i\}, \ n \in \{0, 1, 2, ...\}$$

Definition: transition matrix after n-time steps

$$P^{(n)} = (p_{ij}^{(n)})$$

Transition probability after n-time steps

Definition:

$$p_{ij}^{(0)} = \mathcal{P}\{X_0 = j | X_0 = i\} = \begin{cases} 1 \text{ se } i = j \\ 0 \text{ se } i \neq j \end{cases}$$

$$p_{ij}^{(1)} = \mathcal{P}\{X_1 = j | X_0 = i\} = p_{ij}$$

$$p_{ij}^{(n)} = \mathcal{P}\{X_n = j | X_0 = i\},\ n \in \{0, 1, 2, \ldots\}$$

Properties:

$$0 \leq p_{ij}^{(n)} \leq 1\ \forall i, j \in \mathcal{S}\ \forall n \geq 0$$

$$P^{(0)} = I \qquad P^{(1)} = P.$$

$$\Sigma_{i=0,..,n}\, p_{ij} = 1$$

$$\sum_{j \in \mathcal{S}} p_{ij}^{(n)} = \sum_{j \in \mathcal{S}} \mathcal{P}\{X_n = j | X_0 = i\} = 1\ \forall i \in \mathcal{S}\ \forall n \geq 0$$

It can be proved that:

$P^{(n)} = P^n$ where $P^n = P \cdot P \cdot \ldots \cdot P$

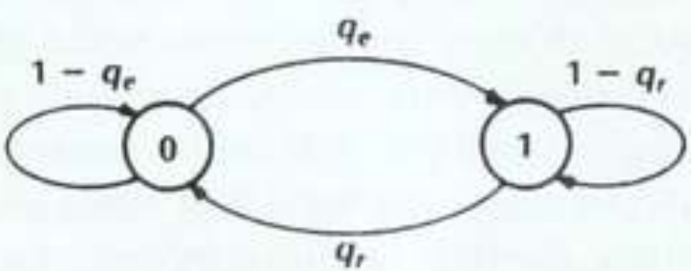
the n-th power of P

Discrete-time Markov model of a single system with repair

- all state transitions occur at fixed intervals
- probabilities assigned to each transition

The probability of state transition depends only on the current state

Graph model



0, 1—States

q_e, q_r — State transition probabilities

Transition Probability Matrix P

$$
\begin{array}{cc}
\text{Current State} & \text{New State} \\
 & \begin{array}{cc} \quad 0 & \qquad 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1 - q_e & q_e \\ q_r & 1 - q_r \end{bmatrix} = P
\end{array}
$$

State 0 : working
State 1: failed

Arcs are possible state transitions

Pij = probability of a transition from state i to state j

- Pij >=0
- the sum of each row must be one

Discrete-time Markov models

Solved by a set of linear equations:

$$[p_0(k+1), p_1(k+1)] = [p_0(k), p_1(k)] \begin{bmatrix} 1-q_e & q_e \\ q_r & 1-q_r \end{bmatrix}$$

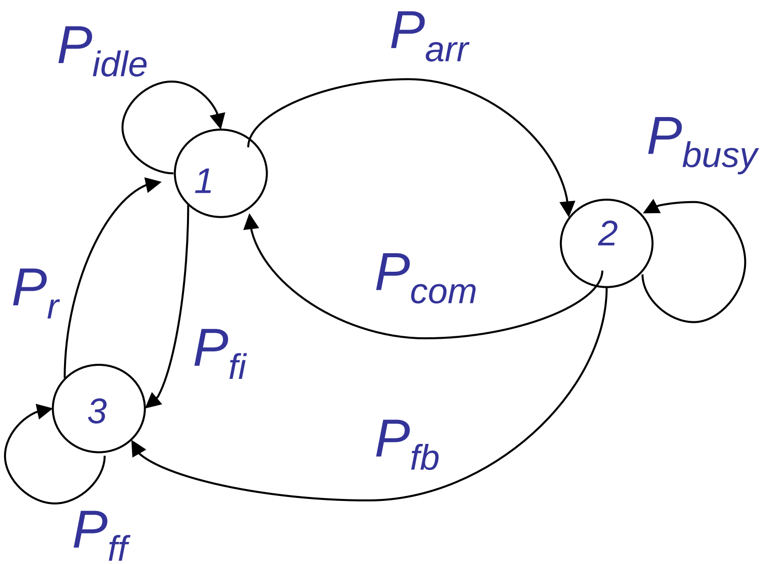
probability of being in state 0 after (k+1) transitions

Probability distribution of a transition from one state i to another state j in no more than k steps:

state j can be made a trapping state with pjj = 1

Another example

A computer is idle, working or failed. When the computer is idle jobs arrives with a given probability. When the computer is idle or busy it may fail with probability P_{fi} or P_{fb}, respectively.



$\{X_t, t>=0\}$: state of the computer at time t

S={1,2,3}
1 computer idle
2 computer working
3 computer failed

$$P = \begin{bmatrix} P_{idle} & P_{arr} & P_{fi} \\ P_{com} & P_{busy} & P_{fb} \\ P_r & 0 & P_{ff} \end{bmatrix}$$

Continuous-time Markov models

Continuous-time models:

state transitions occur at random intervals

transition rates assigned to each transition

The length of time already spent in a state does not influence either the probability distribution of the next state or the probability distribution of remaining time in the same state before the next transition

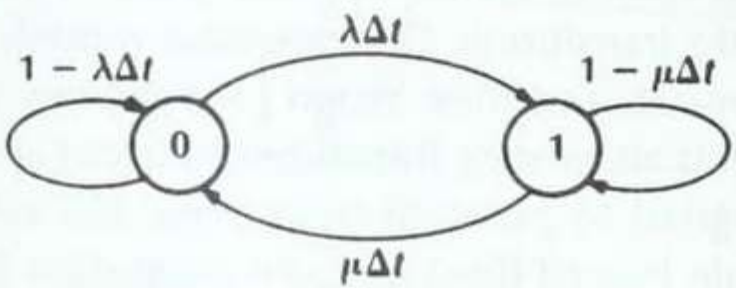
Single system with repair

λ failure rate, μ repair rate

$p_0(t)$ probability of being in the operational state

$p_1(t)$ probability of being in the failed state

Graph model



$\lambda\Delta t$, $\mu\Delta t$—State transition probabilities

λ, μ—State transition rates

Transition Matrix P

$$P = \begin{bmatrix} 1 - \lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1 - \mu\Delta t \end{bmatrix}$$

Continuous-time Markov models

Probability of being in state 0 or 1 at time t+Δt:

$$[p_0(t+\Delta t), p_1(t+\Delta t)] = [p_0(t), p_1(t)] \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

probability of being in state 0 at time t+Δt

Performing multiplication, rearranging and dividing by Δt, taking the limit as Δt approaches to 0:

$$\frac{dp_0(t)}{dt} = \dot{p}_0(t) = -\lambda p_0(t) + \mu p_1(t)$$

$$\frac{dp_1(t)}{dt} = \dot{p}_1(t) = \lambda p_0(t) - \mu p_1(t)$$

Chapman-Kolmogorov equations

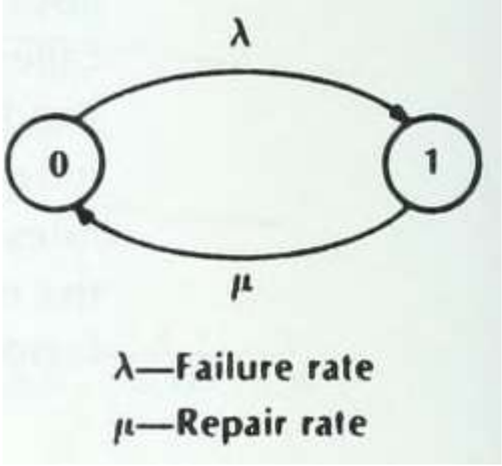
Continuous-time Markov models

Matrix form:

T matrix

$$[\dot{p}_0(t), \dot{p}_1(t)] = [p_0(t), p_1(t)] \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

The set of equations can be written by inspection of a transition diagram without self-loops and Δt's:



Continuous time Markov model graph

The change in state 0 is minus the flow out of state 0 times the probability of being in state 0 at time t, plus the flow into state 0 from state 1 times the probability of being in state 1.

Continuous-time Markov models

Chapman-Kolmogorov equations solved by use of a LaPlace transform of a time domain function

$$[p_0(0),\, p_1(0)] = [p_0^x(s),\, p_1^x(s)] \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}$$

probability of being in state 0 at time t=0

A matrix

$$\vec{P}(0) = \vec{P}^x(s)[s\mathbf{I} - \boldsymbol{T}] = \vec{P}^x(s)A$$

$$\vec{P}^x(s) = \vec{P}(0)[s\mathbf{I} - \boldsymbol{T}]^{-1} = \vec{P}(0)A^{-1}$$

where **I** is the identity matrix

We solve the equations. We obtain as solutions a ratio of two polynomials in s. and we apply the inverse transform to the solutions.

Continuous-time Markov models

Our example

$$A^{-1} = \frac{\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}}{s^2 + \lambda s + \mu s}$$

Assume the system starts in the operational state: P(0) = [1,0]

$$\overline{P}^x(s) = [1,0] \begin{bmatrix} \dfrac{s+\mu}{s^2+\lambda s+\mu s} & \dfrac{\lambda}{s^2+\lambda s+\mu s} \\[2ex] \dfrac{\mu}{s^2+\lambda s+\mu s} & \dfrac{s+\lambda}{s^2+\lambda s+\mu s} \end{bmatrix}$$

$$p_0^x(s) = \frac{s+\mu}{s^2+\lambda s+\mu s}$$

$$p_1^x(s) = \frac{\lambda}{s^2+\lambda s+\mu s}$$

We apply the inverse transforms.

Continuous-time Markov models

$$p_0(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$p_1(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

⇦ A(t)

p_0(t) probability that the system is in the operational state at time t, availability at time t

The availability consists of a steady-state term and an exponential decaying transient term

Only steady-state solution

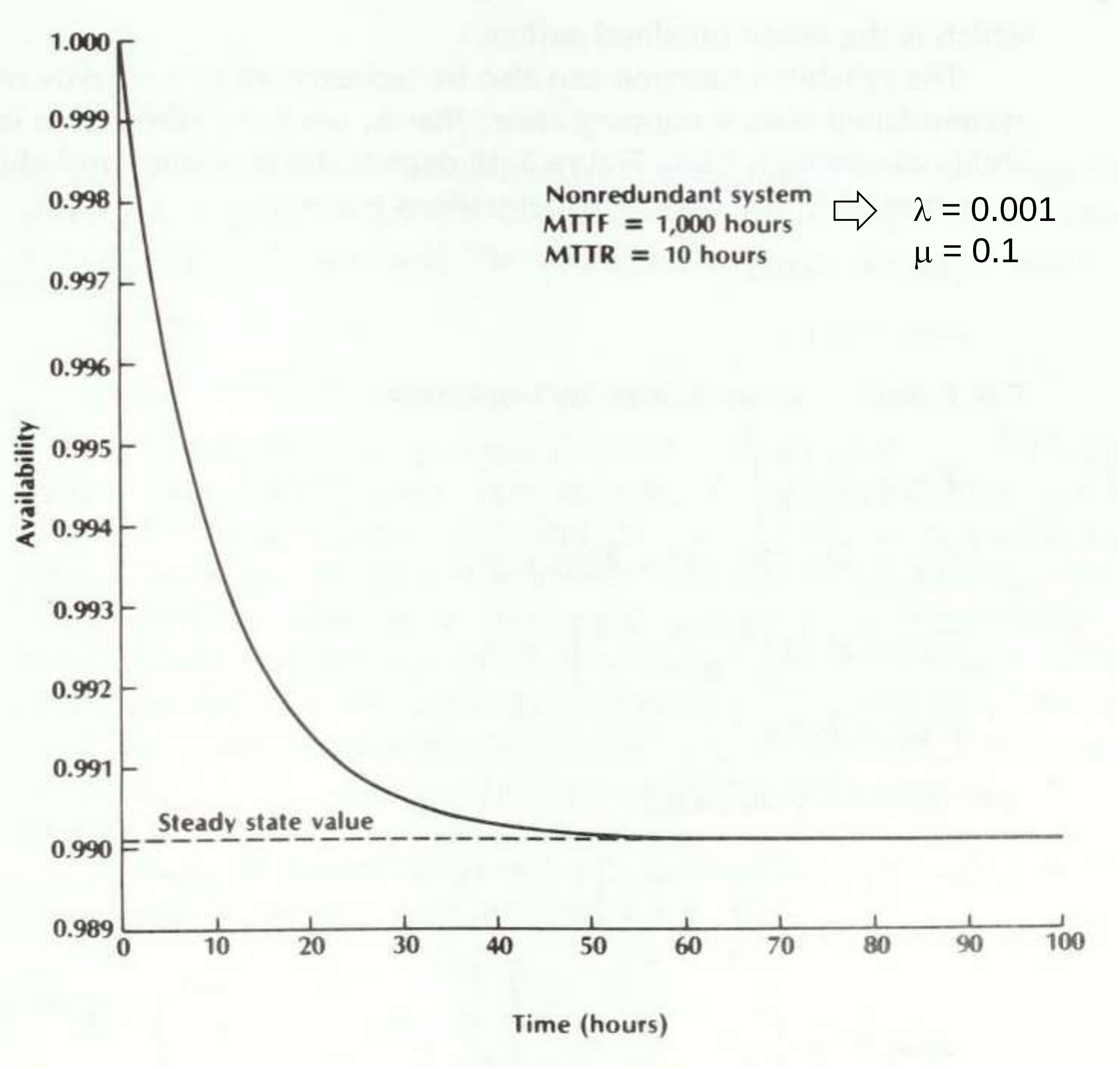
Chapman-Kolmogorov equations: derivative replaced by 0; p0(t) replaced by p0(0) and p1(t) replaced by p1(0)

$$0 = -\lambda p_0 + \mu p_1$$
$$0 = \lambda p_0 - \mu p_1$$

⇨

$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu}} = \frac{\mu}{\lambda + \mu}$$

Availability as a function of time



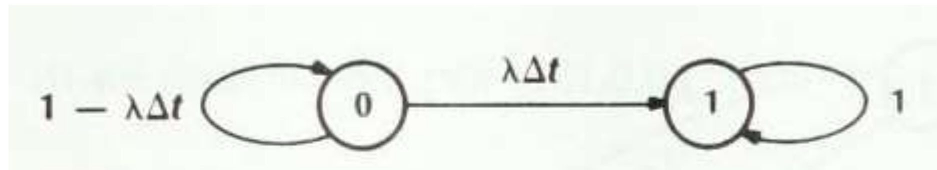
$\lambda = 0.001$
$\mu = 0.1$

The steady-state value is reached in a very short time

Continuous-time Markov models: Reliability

system-failed state a trapping state

Single system without repair



λΔt = state transition probability

Differential equations:

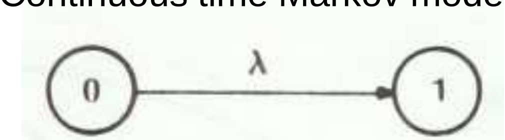
$$\dot{p}_0(t) = -\lambda p_0(t)$$

$$\dot{p}_1(t) = \lambda p_0(t)$$

T matrix

$$\begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$

Continuous time Markov model graph



λ = failure rate

Continuous-time Markov models: Reliability

A matrix
A= [sI –T]

$$\begin{bmatrix} s+\lambda & -\lambda \\ 0 & s \end{bmatrix}$$

$$\vec{P}^x(s) = \vec{P}(0)A^{-1}$$

$$\vec{P}^x(s) = [1,0]\,\frac{\begin{bmatrix} s & \lambda \\ 0 & s+\lambda \end{bmatrix}}{s^2+\lambda s}$$

$$p_0^x(s) = \frac{1}{s+\lambda}$$

$$p_1^x(s) = \frac{1}{s} - \frac{1}{s+\lambda}$$

Taking the inverse transform:

$$p_0(t) = e^{-\lambda t}$$

$$p_1(t) = 1 - e^{-\lambda t}$$

Markov chain

A **Markov chain** is a Markov process X with *discrete-state space S*.

A Markov chain is h*omogeneous* if X has steady-state transition probabilities

A Markov chain is a *finite-state Markov chain* if the number of states is finite (N).

Discrete-time homogeneous Markov chains (DTMC)

Continuous-time homogeneous Markov chains (CTMC)