PECSN, 3rd appello 2025

Exercise 1

Consider the following JPDF of RVs X, Y

$$f(x,y) = \begin{cases} \lambda^2 \cdot e^{-(\lambda x+\lambda y)} \cdot e^{-\alpha xy} & x, y > 0 \\ 0 & otherwise \end{cases}$$

With $\lambda > 0, \alpha \geq 0$.

1) Compute the PDF of X and Y and compute the value of α such that they are independent

For the above value for α and $\lambda = 8$:

2) Compute the PDF of $W = \frac{X+Y}{X}$
3) Compute $E[W-1]$
4) Compute $P\{\frac{1}{2} \leq W \leq 3/4\}$
5) What type of RV is W?

Exercise 2

A buffer has two available slots. Packets arrive with a rate λ to the buffer. The buffer has a service rate equal to μ, and serves packets only when it is full. The buffer can serve either two or one packet simultaneously, with a probability α and $1-\alpha$, respectively.

1) Model the system and draw the CTMC
2) Compute the steady state probabilities.
3) Compute the probability that:
    a. the buffer is empty;
    b. a packet is rejected.

    Check your answer against limit cases.

4) Compute the throughput. Find the minimum and maximum throughput as a function of α.
5) Compute the mean response time of a packet.

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Exercise 1 – solution

1) We obtain:

$$f_X(x) = \int_0^{+\infty} \lambda^2 \cdot e^{-(\lambda x+\lambda y)} \cdot e^{-\alpha xy}\ dy = \lambda^2 \cdot e^{-\lambda x} \int_0^{+\infty} e^{-y(\lambda+\alpha x)} dy = \frac{\lambda^2 \cdot e^{-\lambda x}}{\lambda + \alpha x}$$

Similarly,

$$f_Y(y) = \int_0^{+\infty} \lambda^2 \cdot e^{-(\lambda x+\lambda y)} \cdot e^{-\alpha xy}\ dx = \cdots = \frac{\lambda^2 \cdot e^{-\lambda y}}{\lambda + \alpha y}$$

Independence translates to:

$$f_X(x) \cdot f_Y(y) = f(x, y)$$

$$\frac{\lambda^2 \cdot e^{-\lambda x}}{\lambda + \alpha x} \cdot \frac{\lambda^2 \cdot e^{-\lambda y}}{\lambda + \alpha y} = \lambda^2 \cdot e^{-(\lambda x+\lambda y)} \cdot e^{-\alpha xy}$$

Where the only possibility is $\alpha = 0$. In this case, X, Y are IID exponential RVs.

2) The support for RV $W = \frac{X+Y}{X} = 1 + Y/X$ is $(1, +\infty)$

$$\begin{aligned}
F_W(w) &= P\{W \le w\} = P\{Y \le (w-1) \cdot X\} \\
&= \int_0^{+\infty} \int_0^{(w-1)x} \lambda^2 e^{-\lambda(x+y)} dy dx \\
&= \int_0^{+\infty} \lambda\ \ e^{-\lambda x} \left[\int_0^{(w-1)x} \lambda\ \ e^{-\lambda y} dy \right] dx \\
&= \int_0^{+\infty} \lambda\ \ e^{-\lambda x} \left[1 - e^{-\lambda(w-1)x} \right] dx \\
&= \int_0^{+\infty} \lambda\ \ e^{-\lambda x} dx - \int_0^{+\infty} \lambda\ \ e^{-\lambda w x} dx \\
&= 1 - \frac{1}{w} \int_0^{+\infty} \lambda w \cdot e^{-\lambda w x} dx = 1 - \frac{1}{w}
\end{aligned}$$

From the above, we get $f_W(w) = \frac{1}{w^2}$

3) We get:

$$E[W-1] = E[W] - 1 = \int_{10}^{+\infty} w \cdot \frac{1}{w^2} dw = +\infty$$

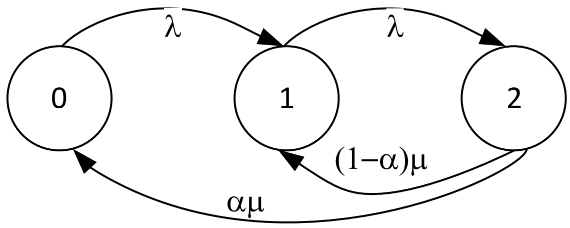
4) The requested probability is zero, since both the endpoints of the interval are outside the support of RV W.

5) W is a heavy-tailed RV. In fact, it is

$$\forall \alpha > 0, \lim_{w \to +\infty} e^{\alpha w} \cdot (1 - F_W(w)) = +\infty$$

Exercise 2

The CTMC is the following:



Global SS equations read:

$$p_0\lambda = p_2\alpha\mu$$
$$p_1\lambda = p_0\lambda + p_2(1-\alpha)\mu$$
$$p_2\mu = p_1\lambda$$

Call $u = \lambda/\mu$ for ease of reading. From the above one obtains:

$$p_0 = \frac{\alpha}{\alpha+1+u} \qquad p_1 = \frac{1}{\alpha+1+u} \qquad p_2 = \frac{u}{\alpha+1+u}$$

The buffer can get empty only if $\alpha > 0$. In this case, it is

$$p_{empty} = p_0 = \frac{\alpha}{\alpha+1+u}$$

Since the (outside) arrival rate is constant, the probability that a packet is refused is:

$$p_L = p_2 = \frac{u}{\alpha+1+u}$$

p_L decreases with α, which makes sense since a larger α makes the buffer empty faster.

The throughput is equal to

$$\gamma = \lambda(1-P_L) = \lambda\frac{\alpha+1}{\alpha+1+u}$$

A more convoluted way to compute it is the following: the system works only when in state 2, and it can serve either one packet or two, at a rate μ with different probabilities. Therefore:

$$\begin{aligned}\gamma &= 2\cdot p_2\cdot\alpha\mu + 1\cdot p_2\cdot(1-\alpha)\mu \\ &= \frac{2\alpha\mu u}{\alpha+1+u} + \frac{(1-\alpha)\mu u}{\alpha+1+u} = \frac{\mu u}{\alpha+1+u}\cdot(\alpha+1) \\ &= \lambda\frac{\alpha+1}{\alpha+1+u}\end{aligned}$$

The throughput increases with α, from a minimum of $\frac{\lambda}{1+u}$ to a maximum of $\frac{2\lambda}{2+u}$.

The mean response time of a packet can be computed via Little's theorem, as:

$$E[R] = \frac{E[N]}{\gamma} = \left(\frac{1}{\alpha+1+u} + \frac{2u}{\alpha+1+u}\right)\cdot\frac{1}{\lambda}\cdot\frac{\alpha+1+u}{\alpha+1} = \frac{1}{\lambda}\cdot\frac{1+2u}{\alpha+1}$$