PECSN, 3/2/2025

Exercise 1

A job consisting of two tasks X and Y has to be scheduled. The duration of the tasks is random, and the JPDF of their duration is:

$$f(x,y) = \begin{cases} x \cdot e^{-(x+y)} & x > 0, y > 0 \\ 0 & otherwise \end{cases}$$

1) A requirement is that both conditions a. and b. stated below must hold simultaneously:
   a. the first task lasts less than the second;
   b. the first task lasts less than 2 seconds.

   What is the probability that the schedule meets the requirements?

2) Is the duration of the two tasks independent? Justify your answer.
3) Compute the mean duration of the whole schedule.
4) Compute the variance of the duration of the whole schedule.

Note: you may want to recall that:

$$\int x \cdot e^{-x} dx = -e^{-x}(x+1) + C$$

$$\int x^2 \cdot e^{-x} dx = -e^{-x}(x^2 + 2x + 2) + C$$

$$\int x^3 \cdot e^{-x} dx = -e^{-x}(x^3 + 3x^2 + 6x + 6) + C$$

Exercise 2

A job may require three different types of processing, respectively called *pre-processing, computation, post-processing*. A new job is first inspected by a *scheduler*, which dispatches it to one of the three processing modules: half of the jobs start with pre-processing, one fourth start with computation, and one fourth start with post-processing. After preprocessing or computation, half of the jobs leave the system, whereas the other half go on to the next processing stage.

There are K jobs are in the system at any time. When a job leaves, a new one is admitted. The system has one processing unit per task. Pre-processing, computation and post-processing take an exponentially distributed time, and its mean is $1/\mu$. The scheduler is instead twice as fast as that.

1) Model the system and compute the routing matrix;

2) Solve the routing equations and compute the SS probabilities in their general form;

3) Compute the normalizing constant as a function of the number of customers K;

4) Compute the utilization of the processing units and the scheduler; check the formula in limit cases;

5) Compute the system throughput.

Exercise 1 – solution

1) The requested probability is

$$P\{X < Y, X < 2\} = \iint_C f(x,y)\,dx\,dy$$

Where $C \equiv \{x > 0; y > 0;\ x \leq 2;\ x < y\}$. Therefore,

$$P\{X < Y, X < 2\} = \int_0^2 \left(\int_x^{+\infty} x \cdot e^{-(x+y)}\,dy \right) dx$$
$$= \int_0^2 x \cdot e^{-x} \left(\int_x^{+\infty} e^{-y}\,dy \right) dx = \int_0^2 x \cdot e^{-2x}\,dx = \frac{1}{2}\int_0^2 2x \cdot e^{-2x}\,dx$$
$$= \frac{1}{2} \cdot \frac{1}{2} \int_0^4 y \cdot e^{-y}\,dy = \frac{1}{4} \cdot [-y \cdot e^{-y} - e^{-y}]_0^4 = \frac{1 - 5e^{-4}}{4}$$

2) One needs to find the individual densities $f_X(x), f_Y(y)$ and check that $f_X(x) \cdot f_Y(y) = f(x,y)$.

$$f_X(x) = \int_0^{+\infty} x \cdot e^{-(x+y)}\,dy = x \cdot e^{-x}$$

$$f_Y(y) = \int_0^{+\infty} x \cdot e^{-(x+y)}\,dx = e^{-y}$$

$f_X(x) \cdot f_Y(y) = f(x,y)$, hence the two RVs are independent.

3) It is $E[X+Y] = E[X] + E[Y]$, therefore

$$E[X] = \int_0^{+\infty} x^2 \cdot e^{-x}\,dx = 2$$

$$E[Y] = \int_0^{+\infty} y \cdot e^{-y}\,dy = 1$$

Both the above need not be computed if one remembers the first two moments of an exponential RV.

Therefore, it is $E[X+Y] = E[X] + E[Y] = 3$

4) Since X, Y are independent, $Var(X+Y) = Var(X) + Var(Y)$

$$E[X^2] = \int_0^{+\infty} x^3 \cdot e^{-x}\,dx = 6$$

$$E[Y^2] = \int_0^{+\infty} y^2 \cdot e^{-y}\,dy = 2$$

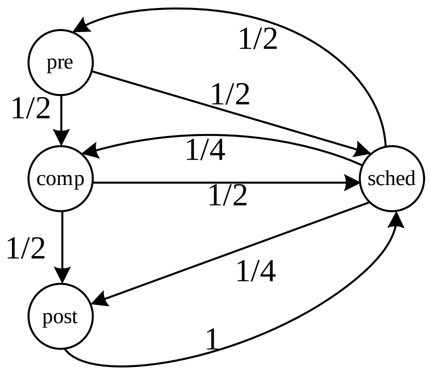
Hence

$$Var(X+Y) = Var(X) + Var(Y) = (6 - 2^2) + (2 - 1^2) = 3$$

Exercise 2 - Solution

The system can be modeled by a closed QN, with four SCs (*pre, comp, post, sched* for short from now on).

The QN diagram and routing matrix are the following:



	pre	comp	post	sched
pre	0	1/2	0	1/2
comp	0	0	1/2	1/2
post	0	0	0	1
sched	1/2	1/4	1/4	0

The input/output balance relationships yield the following:

$$\lambda_{pre} = \lambda_{sched}/2$$

$$\lambda_{comp} = \frac{\lambda_{sched}}{4} + \frac{\lambda_{pre}}{2} = \frac{\lambda_{sched}}{2}$$

$$\lambda_{post} = \frac{\lambda_{sched}}{4} + \frac{\lambda_{comp}}{2} = \frac{\lambda_{sched}}{2}$$

Therefore, a solution to the routing equations is $e^T = [e, e, e, 2e]^T$. Setting $e = \mu$, one finds $\rho^T = [1,1,1,1]^T$.

Therefore, we have $p(n_{pre}, n_{comp}, n_{post}, n_{sched}) = \frac{1}{G(M,K)} \cdot 1^{n_{pre}} \cdot 1^{n_{comp}} \cdot 1^{n_{post}} \cdot 1^{n_{sched}} = \frac{1}{G(M,K)}$

The normalizing constant is the cardinality of set $|\mathcal{E}|$, since all states are equally likely. Therefore, it is $G(M, K) = \binom{K+3}{3}$.

The utilization of the processing elements and the scheduler is the same, and it is equal to $U = \frac{G(4,K-1)}{G(4,K)} = \frac{\binom{K+2}{3}}{\binom{K+3}{3}} = \frac{K}{K+3}$. The expression makes perfect sense, since the utilization grows to 1 with K.

The throughput at the scheduler is: $\gamma = 2\mu \cdot \frac{K}{K+3}$.