Exercise 1

A voice codec alternates between two states (*idle* and *talkspurt*). When in *idle* state, it sends one packet after t_i seconds from the last transmission, and when in *talkspurt* it sends a packet after t_t seconds from the last transmission, $t_t < t_i$. The durations of idle and talkspurt states are exponentially distributed, with mean $1/\lambda_i$ and $1/\lambda_t$ respectively.

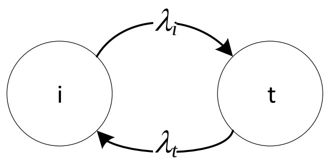
1) Compute the probability that the codec is in the *talkspurt* and *idle* states (assume an infinitely long observation window)
2) Compute the mean generation time of packets at the codec
3) You observed a sequence of n packets spaced t_t units of time. What is the probability that the codec will output the next packet after t_t more units of time? Draw a graph of the above probability as a function of n

Exercise 2

Consider a system having K *gates*, through which job requests may arrive. Through each gate, job requests arrive exponentially at a rate λ. Arrivals processes at different gates are independent. When there are $j < K$ requests in the system, $j + 1$ gates are open, whereas the others are closed. If there are K or more requests, all the K gates are open. The system has K identical servers, with a serving rate equal to μ.

1) Model the above system as a queueing system and draw its CT Markov Chain.
2) Compute the steady-state probabilities and state the stability condition
3) Compute the state that a random observe is more likely to observe and the mean number of jobs in the system
4) Compute the throughput of the system. Compare it to an M/M/1's and discuss the result.
5) Compute the probabilities observed by an arriving job request
6) Compute the mean response time, the mean waiting time and the mean number of queued jobs.

Exercise 1 - Solution



The codec can be seen as a 2-state Markov Chain, with transition rates λ_i and λ_t. Call P_i and P_t the steady-state probabilities of being in the idle and talkspurt states, we readily obtain:

$$P_i \cdot \lambda_i = P_t \cdot \lambda_t$$

$$P_i + P_t = 1$$

From which we obtain $P_i = \frac{\lambda_t}{\lambda_i+\lambda_t}$, $P_t = \frac{\lambda_i}{\lambda_i+\lambda_t}$.

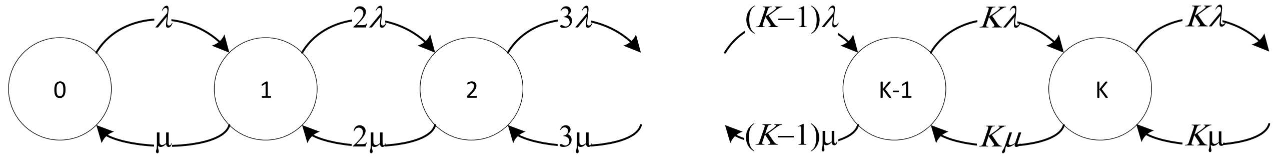
The mean packet generation time is $E[T] = t_i \cdot P_i + t_t \cdot P_t = t_i \cdot \frac{\lambda_t}{\lambda_i+\lambda_t} + t_t \cdot \frac{\lambda_i}{\lambda_i+\lambda_t}$

The CDF is the one in the figure below.

Call X the duration of a talkspurt period. The last question can be formulated as follows: $P\{X \geq (n+1) \cdot t_t | X \geq n \cdot t_t\}$. Since X is exponential, hence memoryless, this is equal to $P\{X \geq t_t\} = e^{-\lambda_t}$, and it is independent of n. The graph is thus a straight line.

Exercise 2 – solution

The CTMC is the following:



From the above (transitions are nearest-neighbor) one straightforwardly obtains $p_j = \left(\frac{\lambda}{\mu}\right)^j \cdot p_0$. Call $\rho = \lambda/\mu$, then the SS probabilities and the stability condition are the same as an M/M/1's, i.e. $p_j = (1-\rho) \cdot \rho^j$, as long as $\rho < 1$.

The state that a random observer is more likely to observe is therefore state 0. The mean number of jobs in the system is given by the Kleinrock function $E[N] = \frac{\rho}{1-\rho}$.

The throughput is

$$\gamma = \sum_{j=1}^{+\infty} j \cdot \mu_j = \sum_{j=1}^{K} j \cdot \mu \cdot (1-\rho) \cdot \rho^j + \sum_{j=K+1}^{+\infty} K \cdot \mu \cdot (1-\rho) \cdot \rho^j$$

$$= \mu \cdot (1-\rho) \sum_{j=1}^{K} j \cdot \rho^j + K \cdot \mu \cdot (1-\rho) \sum_{j=K+1}^{+\infty} \rho^j$$

$$= \mu \cdot (1-\rho)\rho \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2} + K \cdot \mu \cdot (1-\rho)\frac{\rho^{K+1}}{1-\rho}$$

$$= \lambda \cdot \frac{1-(K+1)\rho^K + K\rho^{K+1}}{1-\rho} + K \cdot \mu \cdot \rho^{K+1}$$

$$= \lambda \cdot \left[\frac{1-\rho^K - K\rho^K(1-\rho)}{1-\rho} + K \cdot \rho^K\right]$$

$$= \lambda \cdot \frac{1-\rho^K}{1-\rho}$$

When $K = 1$, this system is an M/M/1, and from the above expression we have $\gamma = \lambda$. When $K > 1$, the throughput is *larger* than an M/M/1's, since the average arrival rate is larger than λ. Note that we always have $\gamma = \bar{\lambda}$.

The probabilities observed by an arriving job are $r_j = \frac{\lambda_j}{\bar{\lambda}} \cdot p_j$. Therefore we have:

$$r_j = \begin{cases} (j+1) \cdot \dfrac{1-\rho}{1-\rho^K}(1-\rho) \cdot \rho^j & j < K \\ K \cdot \dfrac{1-\rho}{1-\rho^K}(1-\rho) \cdot \rho^j & j \geq K \end{cases}$$

The mean response time, by Little's Law, is:

$$E[R] = \frac{E[N]}{\gamma} = \frac{\rho}{1-\rho} \cdot \frac{1}{\lambda} \cdot \frac{1-\rho}{1-\rho^K} = \frac{1}{\mu} \cdot \frac{1}{1-\rho^K}$$

Moreover, we get:

$$E[W] = E[R] - E[t_s] = \frac{1}{\mu} \cdot \frac{\rho^K}{1-\rho^K}$$

And:

$$E[Nq] = \gamma \cdot E[W] = \frac{1}{\mu} \cdot \frac{\rho^K}{1-\rho^K} \cdot \lambda \cdot \frac{1-\rho^K}{1-\rho} = \frac{\rho^{K+1}}{1-\rho}$$