PECSN 24/7/24

Exercise 1

Let X be a continuous RV, such that $P\{X > t\} = e^{-(log3-log2)t}, \; \forall t > 0$.

1) Define RV $Y = \lfloor X \rfloor$. Find the CDF of Y and compute its mean and variance;
2) Let Z be a Geometric RV with $p_Z = 2/3$, independent of Y. Compute $P\{Y = 2, Z < 1\}$ and $P\{Y + Z = 1 | Z < 1\}$;
3) Compute the PMF of $W = \min(Y, Z)$;
4) Compute $P\{Y \geq 3Z\}$.

Exercise 2

In a client-server system, a *client* sends transaction requests to a *server*. The maximum number of outstanding requests at the server is limited by K, and the client stops sending transactions when the server is full. When the server completes a transaction, it sends a response to the client. Transmission of both transaction requests (from the client to the server) and responses (from the server to the client) take an exponential time, with a mean $1/\mu_1$ and $1/\mu_2$, respectively, with $\mu_1 > \mu_2$. Assume that the client always has transaction requests to send to the server, so that the server always has K outstanding requests.

1) Model the above system and compute its steady-state probabilities;
2) Compute the throughput and the utilization of both client and server, as a function of server's transaction limit K;
3) Compute the average time it takes the system to serve a new transaction, as a function of K;
4) Describe what happens to the metrics computed at points 2) and 3) when $\mu_1 \gg \mu_2$.

Exercise 1 – Solution

1) X is an exponential RV with a parameter $\lambda = log3 - log2$. This means that $\lfloor X \rfloor$ is a geometric RV with a parameter $p_Y = 1 - e^{-\lambda} = 1 - e^{-(log3 - log2)} = 1 - e^{log2/3} = 1/3$.

Therefore, it is:

$$P(Y \le k) = 1 - (1 - p_Y)^{k+1} = 1 - \left(\frac{2}{3}\right)^{k+1}, \quad k \ge 0$$

$$E[Y] = \frac{1 - p_Y}{p_Y} = 2$$

$$Var(Y) = \frac{1 - p_Y}{{p_Y}^2} = 6$$

2) Since the two RVs are independent, it is:

$$P\{Y = 2, Z < 1\} = P\{Y = 2\} \cdot P\{Z = 0\} = \left[\left(1 - \frac{1}{3}\right)^2 \cdot \frac{1}{3}\right] \cdot \frac{2}{3} = \frac{4}{27} \cdot \frac{2}{3} = \frac{8}{81}$$

$$P\{Y + Z = 1 | Z < 1\} = \frac{P\{Y + Z = 1\} \cdot P\{Z = 0\}}{P\{Z = 0\}} = P\{Y = 1\} = \left(1 - \frac{1}{3}\right)^1 \cdot \frac{1}{3} = \frac{2}{9}$$

3) The support of W is the same as that of Y, Z, i.e. $k \ge 0$. We have:

$$P\{W > k\} = P\{Y > k, Z > k\} = P\{Y > k\} \cdot P\{Z > k\} =$$

$$= [1 - F_Y(k)] \cdot [1 - F_Z(k)] = \left(\frac{2}{3}\right)^{k+1} \cdot \left(\frac{1}{3}\right)^{k+1} = \left(\frac{2}{9}\right)^{k+1}$$

Therefore, it is $F_W(k) = P\{W \le k\} = 1 - \left(\frac{2}{9}\right)^{k+1}$. W is a geometric RV with a parameter $p_W = \frac{7}{9}$.

Hence, $P\{W = k\} = (1 - p_W)^k \cdot p_W = \left(\frac{2}{9}\right)^k \cdot \frac{7}{9}, \qquad k \ge 0.$

4)We have:

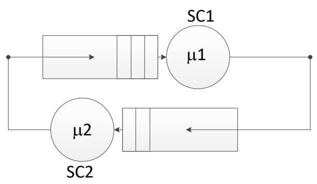
$$P\{Y \ge 3Z\} = \sum_{k=0}^{+\infty} P\{Y \ge 3k, Z = k\} = P\{Y \ge 0\} \cdot P\{Z = 0\} + \sum_{k=1}^{+\infty} P\{Y > 3k - 1\} \cdot P\{Z = k\}$$

$$= p_Z + \sum_{k=1}^{+\infty} (1 - p_Y)^{(3k-1)+1} \cdot [(1 - p_Z)^k \cdot p_Z]$$

$$= \frac{2}{3} + \sum_{k=1}^{+\infty} \left(\frac{2}{3}\right)^{3k} \cdot \left[\left(\frac{1}{3}\right)^k \cdot \frac{2}{3}\right] = \frac{2}{3} \cdot \sum_{k=0}^{+\infty} \left(\frac{8}{81}\right)^k = \frac{2}{3} \cdot \frac{81}{73} = \frac{54}{73}$$

Exercise 2 - Solution

The system can be modeled as a CJN with two SCs and K circulating jobs. Its routing matrix is $\underline{\mathbf{\Pi}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence the solution to the routing equation is $e_1 = e_2$.



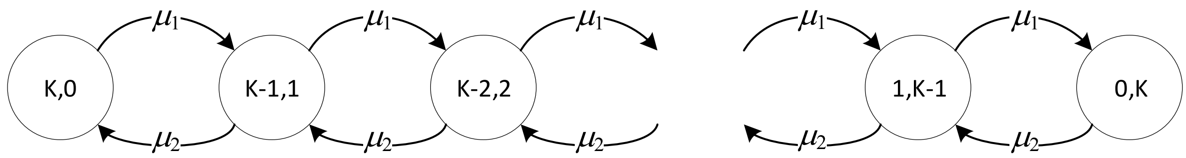
Setting $e_1 = \mu_1$, we get $\rho_1 = 1$, $\rho_2 = \mu_1/\mu_2$. Call $s = \mu_1/\mu_2$, $s > 1$, for simplicity from now on.

By GN's theorem, we get $p(n_1, n_2) = \frac{1}{G(2,K)} \cdot 1^{n_1} \cdot s^{n_2}$, which can be rewritten, having in mind that $n_1 + n_2 = K$, as $p(K-n, n) = \frac{1}{G(2,K)} \cdot s^n$, *n* being the number of jobs in SC2.

In this case, since $\mu_1 \neq \mu_2$, it is $G(2,K) = \frac{s^{K+1}-1}{s-1}$, thus:

$$p(K-n, n) = \frac{s-1}{s^{K+1}-1} \cdot s^n$$

Alternatively, the system can also be modeled via a continuous-time Markov Chain, whose states are $(K - n, n)$, $0 \leq n \leq K$, and whose transition rates are μ_1 (to the right) and μ_2 (to the left). Computing the SS probabilities in the latter yields the selfsame results.



The utilization of the client (SC1) is:

$$U_1 = \rho_1 \cdot \frac{G(M, K-1)}{G(M,K)} = 1 \cdot \frac{1-s^K}{1-s} \cdot \frac{1-s}{1-s^{K+1}} = \frac{s^K - 1}{s^{K+1}-1}$$

Alternatively, one may get to the same results by solving $U_1 = 1 - p(0, K)$.

The utilization of the server (SC2) is:

$$U_2 = \rho_2 \cdot \frac{G(2, K-1)}{G(2,K)} = s \cdot \frac{1-s^K}{1-s^{K+1}} = \frac{s - 1 + 1 - s^{K+1}}{1-s^{K+1}} = 1 - \frac{s-1}{s^{K+1}-1}$$

Or, alternatively, $U_2 = 1 - p(K, 0)$, yielding the same result.

When $K \to +\infty$, U_1 approaches $1/s$, whereas U_2 approaches 1. When $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$), $U_1 \to 0$ and $U_2 \to 1$, since the client is always empty and the sender is always full.

The throughput of both SCs is the same (there is only one path in the CJN) and it is equal to:

$$\gamma_1 = \gamma_2 = \gamma = e_1 \cdot \frac{G(2, K-1)}{G(2,K)} = \mu_1 \cdot \frac{s^K - 1}{s^{K+1}-1} = \mu_2 \cdot \frac{s^{K+1} - s}{s^{K+1}-1}$$

Or, alternatively, $\gamma_1 = \mu_1 \cdot (1 - p(0,K))$, $\gamma_2 = \mu_2 \cdot (1 - p(K, 0))$, yielding the same result.

When $K \to +\infty$ (i.e., when the flow control is ineffective) or $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$), the throughput approaches μ_2, i.e. the slowest of the two systems, from below. This makes perfect sense.

The mean transaction delay is the mean circuit time $E[R] = E[R_1] + E[R_2]$. There are two ways to compute this, one of which is significantly faster. The fast way is to observe that:

$$E[R] = E[R_1] + E[R_2] = \frac{E[N_1]}{\gamma_1} + \frac{E[N_2]}{\gamma_2} = \frac{E[N_1] + E[N_2]}{\gamma} = \frac{K}{\gamma} = \frac{K \cdot (s^{K+1} - 1)}{\mu_1 \cdot (s^K - 1)} = \frac{K}{\mu_2} \cdot \frac{s^{K+1} - 1}{s^{K+1} - s}$$

If one fails to observe straightaway that $E[N_1] + E[N_2] = K$, they can still solve the problem computing $E[N_1]$ and $E[N_2]$ separately (but it will take longer). We have:

$$E[N_i] = \frac{1}{G(M,K)} \cdot \left[\sum_{h=1}^{K} {\rho_i}^h \cdot G(M, K-h)\right]$$

For the client (SC1), the above formula reads:

$$E[N_1] = \frac{s-1}{s^{K+1}-1} \cdot \left[\sum_{h=1}^{K} 1^h \cdot \frac{s^{K+1-h} - 1}{s-1}\right]$$

$$= \frac{1}{s^{K+1}-1} \cdot \sum_{h=1}^{K} [s^h - 1]$$

$$= \frac{1}{s^{K+1}-1} \cdot \left[\frac{1 - s^{K+1}}{1-s} - 1 - K\right]$$

$$= \frac{1}{s-1} - \frac{K+1}{s^{K+1}-1}$$

For the server (SC2), the above formula reads:

$$E[N_2] = \frac{s-1}{s^{K+1}-1} \cdot \left[\sum_{h=1}^{K} s^h \cdot \frac{s^{K+1-h} - 1}{s-1}\right]$$

$$= \frac{1}{s^{K+1}-1} \cdot \sum_{h=1}^{K} [s^{K+1} - s^h]$$

$$= \frac{1}{s^{K+1}-1} \cdot \left[K \cdot s^{K+1} + 1 - \frac{s^{K+1} - 1}{s-1}\right]$$

$$= \frac{K \cdot s^{K+1} + 1}{s^{K+1}-1} - \frac{1}{s-1}$$

And $E[N_1] + E[N_2] = K$, as expected. In any case,

$$E[R] = \frac{K}{\mu_2} \cdot \frac{s^{K+1} - 1}{s^{K+1} - s} = \frac{K}{\mu_2} \cdot \frac{1 - 1/s^{K+1}}{1 - 1/s^K}$$

The latter expression is lower bounded by $\frac{K}{\mu_2}$, which is approached when $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$) or $K \to +\infty$. If the client is considerably faster than the server, all the K jobs are always on the server, and the mean transaction delay is the sum of the service times of all the K jobs that reside there.