Exercise 1

Consider the following function of two variables:

$$f(x,y) = \begin{cases} C \cdot (x+y) & x \in [0,1],\ y \in [0,1], 0 < x + y < 1 \\ 0 & otherwise \end{cases}$$

1) Compute C so that $f(x,y)$ is a JPDF for RVs *X* and *Y*
2) Compute the PDFs for RVs *X* and *Y* and compute $Cov(X,Y)$
3) Define $Z = X + Y$. Compute the PDF of *Z* and determine $E[Z]$

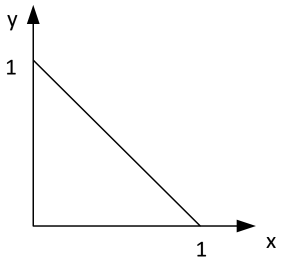
Exercise 2

A server's main job is to handle transactions. These arrive exponentially at a rate λ, they are queued into an infinite FIFO buffer, and they are served at an exponential rate μ. However, transaction handling is sometimes interrupted by *garbage collection (GC) requests*. When one GC request arrives, the sever *stops accepting and serving transactions* and attends to the GC. When the GC request has been cleared, the server resumes accepting/serving transactions. GC requests arrive at a rate γ and are served at a rate δ. However, a GC request may arrive only while the server is handling transactions, and there can only be one outstanding GC request (i.e., GC requests cannot queue up).

1) Model the system and draw the CTMC
2) Write the steady-state equations, find the stability condition and compute the SS probabilities. Explain the stability condition.
3) Express the fraction of time that the server spends handling GC requests.
4) Compute the mean number of transactions in the system.
5) Compute the transaction throughput.
6) Determine what happens of 3,4,5 in the limit cases $\gamma \ll \delta$ and $\gamma \gg \delta$.

Exercise 1 – Solution

The JPDF is non null in the triangle in the figure.



Therefore, normalization reads:

$$\int_0^1 \left[\int_0^{1-x} C(x+y)dy\right] dx = C \cdot \int_0^1 \left[\left[x \cdot y + \frac{y^2}{2}\right]_0^{1-x}\right] dx = C \cdot \int_0^1 \left(\boxed{x} - x^2 + \frac{1+x^2-\boxed{2x}}{2}\right) dx = \frac{C}{2} \cdot \left[x - \frac{x3}{3}\right]_0^1 = \frac{C}{3} = 1$$

Which yields $C = 3$.

2) To obtain the PDF for a RV, we integrate in the other.

$f_X(x) = \int_0^{1-y} 3(x+y)dy = \frac{3}{2}(1-x^2)$, with $x \in [0,1]$.

For obvious reasons of symmetry, $f_Y(y) = \frac{3}{2}(1-y^2)$, with $y \in [0,1]$.

It is $Cov(X,Y) = E[XY] - E[X]E[Y]$. Therefore:

$$E[XY] = \int_0^1\left[\int_0^{1-x} 3xy(x+y)dy\right]dx = 3\int_0^1\left[\int_0^{1-x}\left(x^2 \cdot y + x \cdot y^2\right)dy\right]dx$$

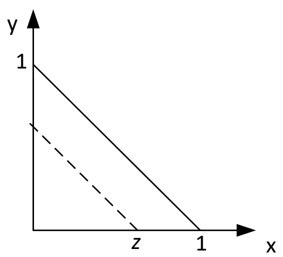
$$= 3\int_0^1\left[x^2 \cdot \frac{1+x^2-2x}{2} + x \cdot \frac{1-3x+3x^2-x^3}{3}\right]dx = \frac{1}{2}\left[x^2 - x^3 + \frac{x^5}{5}\right]_0^1 = \frac{1}{10}$$

$E[X] = E[Y] = \int_0^1 x \cdot \frac{3}{2}(1-x^2)dx = \frac{3}{8}$

Therefore, $Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{10} - \frac{9}{64} = -\frac{13}{320}$

3) By the linearity of the mean value, one can immediately write $E[X] = E[Z] + E[Y] = \frac{6}{8}$.

The PDF of RV Z is non null in [0,1], and $F_Z(z) = P\{X + Y \le z\}$ is the probability that X and Y are in the triangle bounded by the dashed line in the figure.



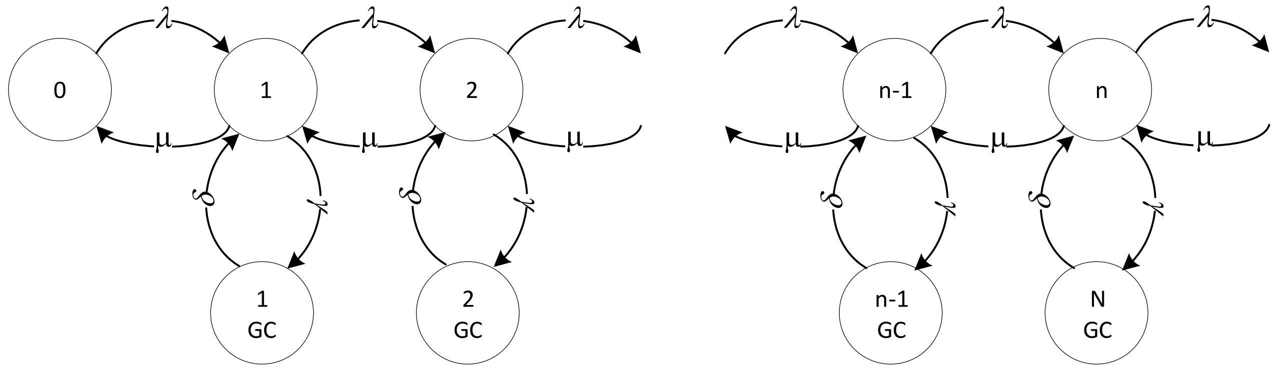
Therefore, computing $F_Z(z)$ only entails substituting z for 1 in the computations already done for the normalization:

$$F_Z(z) = \int_0^z \left[\int_0^{z-x} 3(x+y)dy\right] dx = 3 \cdot \int_0^z \left[\left[x \cdot y + \frac{y^2}{2}\right]_0^{z-x}\right] dx$$
$$= 3 \cdot \int_0^z \left(x \cdot z - x^2 + \frac{z^2 + x^2 - 2z \cdot x}{2}\right) dx = \frac{3}{2} \cdot \left[z^2 \cdot x - \frac{x^3}{3}\right]_0^z = \frac{3}{2} \cdot \frac{2z^3}{3} = z^3$$

From the above, we straightforwardly obtain $f_Z(z) = 3z^2$.

Exercise 2 – solution

The CTMC is the following:



Calling P_j the probabilities of the "upper" states and P_j' those of "lower" states, the SS equations are the following:

$$P_0 \cdot \lambda = P_1 \cdot \mu$$
$$P_j \cdot (\lambda + \mu) = P_{j+1} \cdot \mu + P_{j-1} \cdot \lambda, \quad j > 0$$
$$P_j' \cdot \delta = P_j \cdot \gamma, \ \ j > 0$$

From these, one obtains that $P_j = P_0 \cdot \left(\frac{\lambda}{\mu}\right)^j$, and the following normalization equation:

$$P_0 \cdot \left[1 + \left(1 + \frac{\gamma}{\delta}\right) \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j\right] = 1$$

From which the stability condition is $\lambda < \mu$, and it is independent of λ or δ. This is expectable, since during GC neither arrivals nor services occur, hence the transaction queue does not build up. Calling $\rho = \lambda/\mu$, and $\theta = \lambda/\delta$, we get:

$$P_0 = \frac{1}{1+(1+\theta)\cdot\frac{\rho}{1-\rho}} = \frac{1-\rho}{1+\theta\cdot\rho}, \quad P_j = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \rho^j, \quad P_j' = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \theta \cdot \rho^j$$

The fraction of time that the server spends attending GC requests is

$$P_{GC} = \sum_{j=1}^{+\infty} P_j' = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \theta \cdot \sum_{j=1}^{+\infty} \rho^j = \frac{\theta \cdot \rho}{1+\theta\cdot\rho}$$

The mean number of transactions in the system is

$$E[N] = \sum_{j=1}^{+\infty} j \cdot (P_j + P_j') = \frac{1-\rho}{1+\theta\cdot\rho} \cdot (1+\theta) \cdot \sum_{j=1}^{+\infty} j \cdot \rho^j = \frac{\rho}{1-\rho} \cdot \frac{1+\theta}{1+\theta\cdot\rho}$$

The throughput is:

$$tpt = \sum_{j=1}^{+\infty} \mu_j \cdot P_j = \mu \cdot \frac{1-\rho}{1+\theta\cdot\rho} \cdot \sum_{j=1}^{+\infty} \rho^j = \mu \cdot \frac{1-\rho}{1+\theta\cdot\rho} \cdot \frac{\rho}{1-\rho} = \frac{\lambda}{1+\theta\cdot\rho}$$

The limit cases $\gamma \ll \delta$ and $\gamma \gg \delta$ correspond to $\theta \cong 0$ and $\theta \to \infty$.

a) $\theta \cong 0$: $P_{GC} \cong 0$, $E[N] = \frac{\rho}{1-\rho}$, $tpt = \lambda$. The system behaves like an M/M/1, which is obvious since GCs cause negligible overhead.

b) $\theta \to \infty$: $P_{GC} \to 1$, $E[N] \to \frac{1}{1-\rho}$, $tpt \to 0$. The system throughput goes down to zero. The transaction queue, however, does not grow indefinitely.