Exercise 1

A round-robin scheduling algorithm serves a number of queues cyclically. The algorithm forms *rounds* as follows: it inspects a queue to see if it is *backlogged* or *empty*. If it is empty, it just skips it. If it is backlogged, it extracts one packet and moves on to the next queue. When n packets have been extracted, the algorithm completes the round.

Assume that there are infinitely many queues, and that each queue has a probability p of being backlogged and a probability $1-p$ of being empty when inspected. All queues are independent.

1) Compute the probability $\pi_{j,n}$ that forming a round requires j queue inspections;
2) Compute the mean number of queue inspections that the algorithm has to inspect to complete a round;
3) Compute the variance of the number of queue inspections in a round and the CoV. Discuss what happens when n goes to infinity and justify your answer.

Exercise 2

A firm has M trainees. Each trainee j can serve customers at their own speed, with a service rate μ_j, $1 \leq j \leq M$. The firm adopts the following policy: all trainees follow online courses while there are no customers on the premises. When a customer arrives, the manager selects *one trainee at random* (trainee *j* is chosen with probability π_j), and allocates them to customer service. The selected trainee will serve *all customers*, until there are no more, and then will go back to following online courses. Customers arrive with exponential interarrival times, at a rate λ.

1) Model the above system as a queueing system and draw the CTMC (you can draw the diagram with $M = 2$ for simplicity, and then discuss the generalization).
2) Compute the stability condition and the SS probabilities. Check the result in the limit cases i) $M = 1$, and ii) M identical trainees.
3) Assume that $\pi_j = 1 - \lambda/\mu_j, \quad 1 \leq j \leq M$. Compute the mean number of customers. Provide a physical interpretation for the result.
4) Compute the CDF of the response time of a customer.

Exercise 1 – Solution

1) The probability that the algorithm finds the n-th packet at the j-th inspection is the probability that it finds $n-1$ in $j-1$ inspections (whatever the order), and one at the j-th. This is:

$$\pi_{j,n} = \left[\binom{j-1}{n-1} \cdot (1-p)^{(j-1)-(n-1)} \cdot p^{n-1}\right] \cdot p = \binom{j-1}{n-1} \cdot (1-p)^{j-n} \cdot p^n$$

Where p is the success probability on the single trial, and $j \geq n$. With a little algebra, one gets:

$$\pi_{j,n} = \frac{n}{j} \cdot \binom{j}{n} \cdot (1-p)^{j-n} \cdot p^n$$

2) Call J the RV counting the number of inspections in a round. The direct method to compute the mean value is:

$$E[J] = \sum_{j=n}^{+\infty} j \cdot \pi_{j,n} = \sum_{j=n}^{+\infty} \left[n \cdot \binom{j}{n} \cdot (1-p)^{j-n} \cdot p^n\right]$$

The above sum may be tricky to solve. A quicker way to solve it is to observe that J is the sum of n IID geometric RVs, each one measuring the number of *trials* required to obtain the first success (hence having a support starting from 1). For such a variable, call it T, we have $p(i) = (1-p)^{i-1} \cdot p$. Therefore, we can compute its PGF as

$$G(z) = \sum_{i=1}^{+\infty} (1-p)^{i-1} \cdot p \cdot z^i = \frac{z \cdot p}{1 - z + z \cdot p}$$

Hence we have:

$$E[T] = G'(1) = \left.\frac{p \cdot (1 - z + z \cdot p) - zp \cdot (-1+p)}{(1 - z + z \cdot p)^2}\right|_{z=1} = \left.\frac{p}{(1 - z + z \cdot p)^2}\right|_{z=1} = \frac{1}{p}$$

For the above reason, the required mean value is $E[J] = n \cdot E[T] = \frac{n}{p}$.

3) Once the PGF has been computed, one may compute

$$Var(J) = n \cdot Var(T) = n \cdot [G''(1) + G'(1) - G'(1)^2]$$

because of independence. The only missing term in the above computation is $G''(1)$, which is:

$$G''(1) = \left.\frac{-p \cdot [2z(p-1)^2 + 2(p-1)]}{(1 - z + z \cdot p)^4}\right|_{z=1} = \frac{2(1-p)}{p^2} = \frac{2}{p^2} - \frac{2}{p}$$

Hence:

$$Var(T) = \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

And $Var(J) = n \cdot \frac{1-p}{p^2}$.

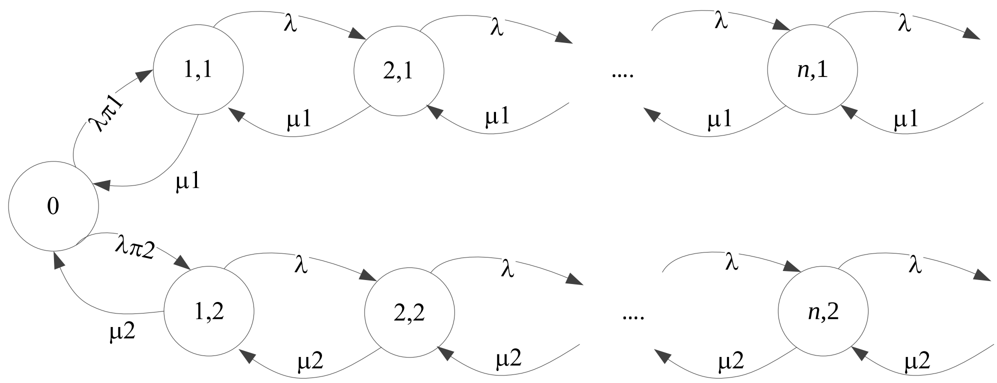
From the above, we obtain:

$$CoV(J) = \frac{std(J)}{E[J]} = \sqrt{n \cdot \frac{1-p}{p^2}} \Big/ \frac{n}{p} = \sqrt{\frac{1-p}{n}}$$

The CoV goes to zero when n increases. Since J is a sum of IID RVs, this is a consequence of the CLT.

Exercise 2 – Solution

1) We represent the state of the system with a couple (n, j), where n is the number of customers and j is the number of the selected trainee. The CTMC diagram (for the case $M = 2$) is the following. The generalization is straightforward (M horizontal branches going to infinity, each one branching out of state 0 with a rate $\lambda \cdot \pi_j$, where service rates are μ_j and arrival rates are λ.



2) Using local equilibrium equations, one can write:

$$p_{n,j} = \left(\frac{\lambda}{\mu_j}\right)^n \cdot \pi_j \cdot p_0 = \rho_j{}^n \cdot \pi_j \cdot p_0, \quad n > 0, 1 \le j \le M$$

From which we get that the stability condition is clearly $\lambda < \min_j(\mu_j)$. Moreover, the following can be written:

$$p_0 \cdot \left[1 + \sum_{j=1}^{M} \pi_j \cdot \sum_{n=1}^{+\infty} \rho_j{}^n\right] = 1$$

Since $\sum_{j=1}^{M} \pi_j = 1$, under the stability condition the previous expression yields:

$$p_0 = \frac{1}{\sum_{j=1}^{M} \frac{\pi_j}{1-\rho_j}}$$

$$p_{n,j} = \rho_j{}^n \cdot \pi_j \cdot \frac{1}{\sum_{j=1}^{M} \frac{\pi_j}{1-\rho_j}}, \qquad n > 0, 1 \le j \le M$$

If $M = 1$ we obtain the same SS probabilities as an M/M/1 system's. The same occurs if the trainees are identical, in which case it is $\rho_j = \rho$, and we get $p_n = \sum_{j=1}^{M} p_{n,j} = (1-\rho) \cdot \rho^n$.

3) When $\pi_j = 1 - \lambda/\mu_j$, it is $\pi_j/(1-\rho_j) = 1$, the sum at the denominator becomes M, and we get $p_{n,j} = {\rho_j}^n \cdot \pi_j/M$. In this case we get:

$$E[N] = \sum_{j=1}^{M}\sum_{n=0}^{+\infty} n \cdot p_{n,j} = \frac{1}{M} \cdot \sum_{j=1}^{M} \pi_j \sum_{n=0}^{+\infty} n \cdot {\rho_j}^n$$

$$= \frac{1}{M} \cdot \sum_{j=1}^{M} \pi_j \frac{\rho_j}{\left(1-\rho_j\right)^2}$$

$$= \frac{1}{M} \cdot \sum_{j=1}^{M} \frac{\rho_j}{1-\rho_j}$$

The above result is the average of the Kleinrock functions of M/M/1 systems each having a particular trainee as a server.

4) The PDF of the response time R can be computed using total probability. When the system is in state (n, j), the response time is an $(n+1)$-stage Erlang with a rate μ_j.

$$f_R(t) = \sum_{j=1}^{M}\sum_{n=0}^{+\infty} f_{E_{n+1,j}}(t) \cdot r_{n,j}$$

Where $r_{n,j} = p_{n,j}$ and $f_{E_{n+1,j}}(t) = \mu_j e^{-\mu_j \cdot t} \cdot \frac{(\mu_j \cdot t)^n}{n!}$. After a few straightforward manipulations, one gets:

$$f_R(t) = \frac{1}{M} \cdot \sum_{j=1}^{M} \frac{1}{E[R_j]} e^{-\frac{t}{E[R_j]}}$$

Where $E[R_j] = 1/(\mu_j - \lambda)$. The above is the average of the PDFs of individual M/M/1 systems, each one having a service rate μ_j. The CDF is therefore

$$F_R(t) = \frac{1}{M} \cdot \sum_{j=1}^{M} \left(1 - e^{-\frac{t}{E[R_j]}}\right)$$