Exercise 1

The power *consumed* and *produced* by a power plant can be modelled as – respectively - two continuous RVs, X, Y. Their JPDF is equal to:

$$f(x,y) = \begin{cases} \dfrac{1}{4 \cdot \sqrt{xy}} & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise \end{cases}$$

1) Compute the PDFs of X and Y, and state whether or not they are independent;
2) Compute $E[X \cdot Y]$
3) Compute the distribution of the ratio of produced vs. consumed power;
4) Compute the probability that the produced power is more than twice the consumed one.

Exercise 2

Consider a distributed system providing service to a pool of N processes. The system has a pool of $M \geq N$ identical peripherals, which are kept idle until some process requests service to them. Every time a process issues a service request, a peripheral is activated and dedicated to that process. Service requests are blocking, i.e. the same process can only have at most an outstanding request at any time. When the service request is fulfilled, the peripheral is returned to the idle pool and the process carries on with its elaborations. Assume that a running process issues service requests with an interarrival time which is exponentially distributed, and let λ be its rate. Assume that all processes are independent. Assume that the service time of peripherals is exponentially distributed, and let μ be its rate. Assume $\mu \neq \lambda$ for simplicity.

1) Model the system and draw its CTMC.
2) Compute the stability condition and find the steady-state probabilities.
3) Compute the mean number of blocked processes.
4) Compute the steady-state probabilities seen by a process issuing a service request.
5) Compute the CDF of the response time for a process.

Exercise 1 - Solution

1) It is:

$$f_X(x) = \int_0^1 f(x,y)dy = \frac{1}{4\cdot\sqrt{x}}\int_0^1 y^{-\frac{1}{2}}dy = \frac{1}{4\cdot\sqrt{x}}\cdot 2\cdot(1-0) = \frac{1}{2\cdot\sqrt{x}}$$

$$f_Y(y) = \int_0^1 f(x,y)dx = \cdots = \frac{1}{2\cdot\sqrt{y}}$$

The two RVs are independent since $f_X(x)\cdot f_Y(y) = f(x,y)$.

2) From the theory, we know that

$$Cov(X,Y) = E[X\cdot Y] - \mu_X\cdot\mu_Y$$

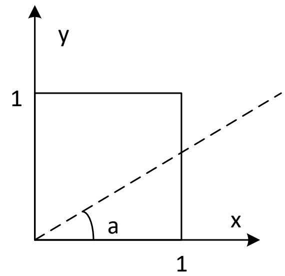
However, since the two RVs are independent, it is $Cov(X,Y) = 0$, hence $E[X\cdot Y] = \mu_X\cdot\mu_Y = \mu_X^2$ since $\mu_Y = \mu_X$, by symmetry.

$$\mu_X = \int_0^1 x\cdot f_X(x)dx = \int_0^1 x\cdot\frac{1}{2\cdot\sqrt{x}}dx = \int_0^1 \frac{1}{2}\cdot x^{\frac{1}{2}}dx = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

Therefore, we have:

$$E[X\cdot Y] = \frac{1}{9}$$

3) We need to compute the CDF of RV $Z = Y/X$. Since both X, Y have support in $[0,1]$, the support of $Z\ is\ [0,+\infty]$. Moreover $F_Z(a) = P\{Z \le a\} = P\{Y \le a\cdot X\}$. The latter is the integral of $f(x,y)$ in the area below the dotted line in the Cartesian plane. Depending on whether $a \le 1, a \ge 1$, the area is a triangle or a trapezoid, hence computations are different.



$$F_Z(a) = \begin{cases} \int_0^1 \left[\int_0^{ax} \frac{1}{4\cdot\sqrt{xy}}dy\right] dx = \frac{1}{2}\sqrt{a} & a \le 1 \\ 1 - \int_0^1 \left[\int_0^{\frac{y}{a}} \frac{1}{4\cdot\sqrt{xy}}dx\right] dy = 1 - \frac{1}{2\sqrt{a}} & a \ge 1 \end{cases}$$

The alert reader can check that $F_Z(a)$ verifies all the conditions of a CDF.

4) The requested probability is $P\{Y \ge 2\cdot X\}$, i.e. $1 - F_Z(2) = 1 - \left(1 - \frac{1}{2\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}$.

Exercise 2 – Solution

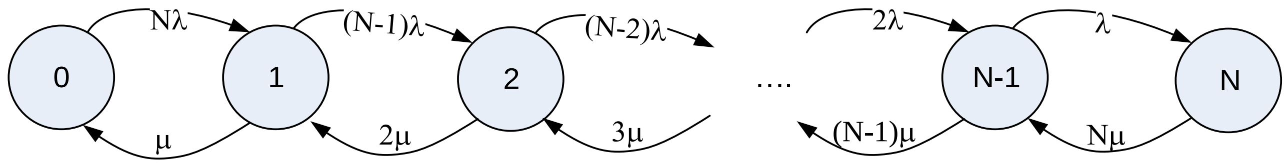
The quicker way. The system is indistinguishable from one where each process has a dedicated peripheral (since there are more peripherals than processes). The latter is the juxtaposition of N independent and non interacting 2-state M/M/1/1 subsystems. Define $u = \frac{\lambda}{\mu}$. The probability that a subsystem like this is empty is $\pi_0 = \frac{1}{(1+u)}$, and the probability that it is non empty is $\pi_1 = \frac{u}{(1+u)}$. Then the probability that a subsystem is busy is Bernoullian, hence the requested probabilities p_j for the whole system can be computed through the binomial formula as:

$$p_j = \binom{N}{j}\cdot{\pi_1}^j\cdot{\pi_0}^{N-j} = \binom{N}{j}\cdot\frac{u^j}{(1+u)^N},$$

And the mean value is $E[n] = N \cdot \pi_1 = N \cdot \frac{u}{(1+u)}$. The system is always stable, since the population is finite. Moreover, there is never any queueing, hence the response time is equal to the service time: $F_R(t) = 1 - e^{-\mu \cdot t}$. Therefore, the only thing that you need to compute from scratch is the requirement for point 4.

The longer way.

1) The system is a finite-population one. The service rate increases based on the number of jobs in the system, and there is never any queueing (an idle server is always available for an incoming job request).



This allows one to answer point 5 immediately: the distribution of the response time is the distribution of the service time, i.e. $F_R(t) = 1 - e^{-\mu \cdot t}$

2) The system is always stable, since there is never any queueing and the number of states is finite. The SS probabilities can be easily found by observing that the CTMC diagram has only nearest-neighbor transitions:

$$p_j = \frac{p_0 \cdot \left(\frac{N!}{(N-j)!} \cdot \lambda^j \right)}{(\mu^j j!)} = \binom{N}{j} \cdot u^j \cdot p_0$$

This said, normalization reads:

$$\sum_{j=0}^{N} p_j = p_0 \cdot \sum_{j=0}^{N} \binom{N}{j} u^j = p_0 \cdot (1+u)^N = 1$$

hence

$$p_j = \binom{N}{j} \cdot \frac{u^j}{(1+u)^N}, \quad 0 \le j \le N$$

3) The mean number of blocked processes is:

$$E[n] = \sum_{j=0}^{N} j \cdot \binom{N}{j} \frac{u^j}{(1+u)^N} = N \cdot \sum_{j=1}^{N} \binom{N-1}{j-1} \frac{u^j}{(1+u)^N} = N \cdot \frac{u}{1+u}$$

4) The system is non-PASTA, since the arrival rates are not constant. It is:

$$\overline{\lambda} = \sum_{j=0}^{N} \lambda_j \cdot p_j = \sum_{j=0}^{N} (N-j) \cdot \lambda \cdot p_j = \left[N \cdot \lambda \cdot \sum_{j=0}^{N} p_j - \lambda \cdot \sum_{j=0}^{N} j \cdot p_j \right] = \lambda \cdot \left[N - E[n] \right]$$

Therefore, we have:

$$r_j = \frac{(N-j) \cdot \lambda}{\lambda \cdot \left[N - E[n] \right]} \cdot p_j = \frac{N-j}{N-E[n]} \cdot p_j = [\dots] = \binom{N-1}{j} \cdot \frac{u^j}{(1+u)^{N-1}}, \quad 0 \le j \le N-1$$

5) See point 1.