PECSN, 17/01/24

Exercise 1

A box contains a number of coins. Of these, some (a fraction $\alpha \leq 1$) are *fair*, whereas for the remaining ones the probability of *heads* is $p \neq 1/2$.

Assume you pick a coin *at random* from the box, and you flip it 100 times.

1) What is the probability that the first flip yields *heads*?
2) Let X be a RV that counts the total number of *heads* in 100 flips. What is its PMF?
3) Compute $E[X]$.
4) Compute α as a function of p when $E[X] = 25$. Discuss your findings.
5) Assume you obtain 50 heads in 100 flips. Compute the probability that your coin is a fair one.

Exercise 2

The output interface of a network router serves packets at a constant speed. Packets have a transmission time which is exponentially distributed, with a rate μ. Packets arrive at the interface with exponential interarrival times. However, the rate of arrivals depends on the number of packets already in the system, with the following law:

$$\lambda_n = \begin{cases} \dfrac{N-n}{N \cdot (n+1)} & n < N \\ 0 & n \geq N \end{cases}$$

1) Model the system and draw its CTMC
2) Compute the stability condition and the steady-state probabilities [*suggestion*: *spend a little time tweaking the SSPs, it will pay off later on*]
3) Compute the SS probabilities seen by an arriving packet
4) Compute the throughput
5) Compute the mean number of packets at the interface and the mean response time of the latter.

Some formulas you may find useful:

- $k\binom{N}{k} = N\binom{N-1}{k-1}$
- $\binom{N-j}{j+1}\binom{N}{j} = \binom{N}{j+1}$
- $\sum_{k=0}^{N}\binom{N}{k} \cdot \frac{1}{x^k} = \left(1+\frac{1}{x}\right)^N$

Exercise 1 – solution

1) According to the law of total probability, we have:

$$P\{heads\} = P\{heads|fair\} \cdot P\{fair\} + P\{heads|unfair\} \cdot P\{unfair\} = \frac{1}{2} \cdot \alpha + p \cdot (1 - \alpha)$$

2) By the same token,

$$\begin{aligned} p_k = P\{X = k\} &= P\{X = k|fair\} \cdot P\{fair\} + P\{X = k|unfair\} \cdot P\{unfair\} \\ &= \binom{100}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{100-k} \cdot \alpha + \binom{100}{k} p^k \cdot (1-p)^{100-k} \cdot (1-\alpha) \\ &= \binom{100}{k} \left[\left(\frac{1}{2^{100}}\right) \cdot \alpha + p^k \cdot (1-p)^{100-k} \cdot (1-\alpha)\right] \end{aligned}$$

3) $E[X] = \sum_{k=0}^{100} k \cdot p_k = \sum_{k=0}^{100} k \cdot [P\{X = k|fair\} \cdot P\{fair\} + P\{X = k|unfair\} \cdot P\{unfair\}]$

Therefore,

$$E[X] = \alpha \cdot 100 \cdot \frac{1}{2} + (1 - \alpha) \cdot 100 \cdot p = 100 \left[\alpha \cdot \left(\frac{1}{2} - p\right) + p\right]$$

4) From the above, we have:

$$100 \left[\alpha \cdot \left(\frac{1}{2} - p\right) + p\right] = 25$$

i.e.,

$$\alpha = \frac{1 - 4p}{2 - 4p}$$

Since $\alpha, p \in [0; 1]$, the above has solutions only when $p \in [0; \frac{1}{4}]$, which makes sense (you cannot have $E[X] = 25$ if *all* your coins have $p > \frac{1}{4}$). When $p \in [0; \frac{1}{4}]$, α is decreasing. When $p = 0$ it is $\alpha = \frac{1}{2}$, which is intuitively correct, and when $p = \frac{1}{4}$ it is $\alpha = 0$, which is also intuitively correct.

5) By Bayes' theorem:

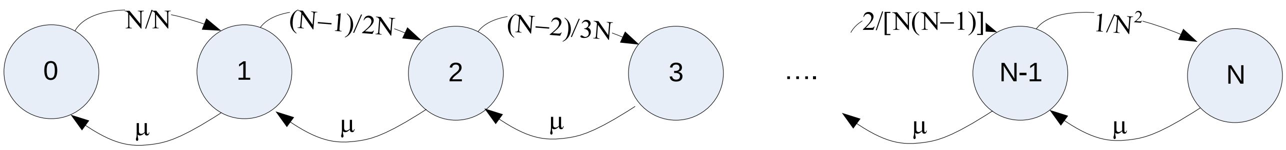
$$P\{fair|X = 50\} = \frac{P\{X = 50|fair\} \cdot P\{fair\}}{P\{X = 50\}}$$

$$\begin{aligned} \frac{\binom{100}{50}\left[\left(\frac{1}{2^{100}}\right) \cdot \alpha\right]}{\binom{100}{50}\left[\left(\frac{1}{2^{100}}\right) \cdot \alpha + p^{50} \cdot (1-p)^{50} \cdot (1-\alpha)\right]} &= \frac{\left(\frac{1}{2^{100}}\right) \cdot \alpha}{\left(\frac{1}{2^{100}}\right) \cdot \alpha + p^{50} \cdot (1-p)^{50} \cdot (1-\alpha)} \\ &= \frac{1}{1 + [4p(1-p)]^{50} \cdot \frac{1-\alpha}{\alpha}} \end{aligned}$$

The above probability is significantly different from 1 only when the second addendum at the denominator is nonnegligible, which happens if $p \cong \frac{1}{2}$ or $\alpha \cong 0$. This matches intuition.

Exercise 2 – solution

The CTMC is the following:



The system is stable, since it has a finite number of states.
Since the transitions are nearest-neighbor, we can write the usual equation that connects p_k to p_0:

$$p_k = \prod_{j=0}^{k-1} \frac{\lambda_j}{\mu_{j+1}} \cdot p_0 = \frac{1}{\mu^k} \cdot \prod_{j=0}^{k-1} \frac{N-j}{N \cdot (j+1)} \cdot p_0 = \binom{N}{k} \cdot \frac{1}{(N\mu)^k} \cdot p_0$$

Which holds for $0 \le k \le N$. Therefore, we can write normalization as follows:

$$\sum_{k=0}^{N} \binom{N}{k} \cdot \frac{1}{(N\mu)^k} \cdot p_0 = 1$$

With the help of one of the suggested formulas, we obtain:

$$p_0 = \frac{1}{\left(1 + \frac{1}{N\mu}\right)^N}$$

Therefore, we have:

$$p_k = \binom{N}{k} \cdot \frac{1}{(N\mu)^k} \cdot \frac{1}{\left(1 + \frac{1}{N\mu}\right)^N} = \binom{N}{k} \cdot \left(\frac{1}{N\mu}\right)^k \cdot \left(\frac{N\mu}{1+N\mu}\right)^N$$

$$= \binom{N}{k} \cdot \left(\frac{1}{N\mu}\right)^k \cdot \left(\frac{N\mu}{1+N\mu}\right)^{N-k} \cdot \left(\frac{N\mu}{1+N\mu}\right)^k = \binom{N}{k} \cdot \left(\frac{1}{1+N\mu}\right)^k \cdot \left(\frac{N\mu}{1+N\mu}\right)^{N-k}$$

Calling $\alpha = \frac{1}{1+N\mu}$, we have $p_k = \binom{N}{k} \alpha^k \cdot (1-\alpha)^{N-k}$.

This system is non-PASTA. Therefore, we expect $r_k \neq p_k$. It is, in fact, $r_k = \frac{\lambda_k}{\bar{\lambda}} \cdot p_k$. Computing $\bar{\lambda}$ is tedious, unless one observes that $\bar{\lambda} = \gamma = \mu(1-p_0)$. Given the above, we have:

$$r_k = \frac{N-k}{N \cdot (k+1)} \cdot \frac{1}{\mu(1-p_0)} \cdot p_k = \frac{N-k}{N \cdot (k+1)} \cdot \frac{1}{\mu(1-(1-\alpha)^N)} \cdot \binom{N}{k} \alpha^k \cdot (1-\alpha)^{N-k}$$

$$= \frac{1}{N\mu} \cdot \binom{N}{k+1} \alpha^k \cdot (1-\alpha)^{N-k} \cdot \frac{1}{(1-(1-\alpha)^N)}$$

Moreover, we have: $\gamma = \mu(1-(1-\alpha)^N)$

The mean number of packets at the interface is the mean of a binomial distribution with a probability of success equal to α, i.e.

$$E[N] = N \cdot \alpha = \frac{N}{1+N\mu} = \frac{1-\alpha}{\mu}$$

By Little's Law, we get that

$$E[R] = \frac{E[N]}{\gamma} = \frac{1-\alpha}{\mu} \cdot \frac{1}{\mu(1-(1-\alpha)^N)} = \frac{1-\alpha}{\mu^2(1-(1-\alpha)^N)}$$