PECSN 20/9/23

Exercise 1

Let $X_j, j = 1,2, \ldots$ be IID RVs, with $E\left[X_j\right] = 0$ and $Var\left(X_j\right) = 2$;

1) Assume that X_j can only take two values, call them a, b. Find $a \cdot b$. Assume then that the two values are equally likely, and find *both* a and b;
2) Assume *instead* that X_j has the following PDF: $f(x) = C \cdot e^{-\lambda|x|}$. Compute C and λ;
3) Estimate $P\left\{\sum_{j=1}^{100} X_j > 10\right\}$ in both cases 1) and 2);
4) Assuming that X_j can only take two equally likely values, compute $P\{X_1 + X_2 = 0\}$. Generalize the above formula to $P\left\{\sum_{j=1}^{2n} X_j = 0\right\}$, for any positive n;
5) Still in the hypotheses of point 4), find the PMF of $S_n = \sum_{j=1}^{2n} X_j$.

Exercise 2

In a publish-subscribe system, subscribers issue *subscribe requests* to a publisher, and then wait for the corresponding notification. Subscribe requests arrive at exponential interarrival times with a rate λ. The publisher publishes *notifications* with a rate μ. All waiting subscribers receive a notification simultaneously, they all leave the system and they have to subscribe again to receive another notification.

1) Draw the CTMC;
2) Find the global equilibrium equations;
3) Compute the conditions according to which the number of waiting requests stays finite, and the PMF of the number of subscribe requests waiting;
4) Find the mean response time for a subscriber;
5) Compute the throughput, i.e., the number of subscribers notified per unit of time.

Exercise 1 – solution

1) The values must satisfy two constraints, one for the mean and one for the variance. Let p be the probability of value a. These constraints are:

$$\begin{cases} a \cdot p + b \cdot (1-p) = 0 \\ a^2 \cdot p + b^2 \cdot (1-p) = 2 \end{cases}$$

From the above equations, we easily obtain $a \cdot b = -2$. If the two values are equally likely, it must be $b = -a$, hence $a = \sqrt{2}$.

2) We need to impose that $\int_{-\infty}^{+\infty} f(x) = 1$. Since the PDF is symmetric around 0, we have $2C \cdot \int_0^{+\infty} e^{-\lambda \cdot x} = 1$, hence $\frac{2C}{\lambda} = 1$, hence $C = \frac{\lambda}{2}$. Moreover, we have $Var(X_j) = E[X_j{}^2] - E[X_j]^2 = E[X_j{}^2] = \int_{-\infty}^{+\infty} x^2 f(x) = 2$. From this we derive $\int_{-\infty}^{+\infty} \frac{\lambda}{2} x^2 e^{-\lambda \cdot |x|} dx = 2 \int_0^{+\infty} \frac{\lambda}{2} x^2 e^{-\lambda \cdot x} dx = 2$. However, the latter is the expression of the mean square value of an exponential RV with a rate λ. For the latter (call it *Y*), we know that $E[Y^2] = Var(Y) + E[Y]^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$. Hence, we have $\frac{2}{\lambda^2} = 2$, or $\lambda = 1$ (λ must in fact be positive), and $C = \frac{1}{2}$.

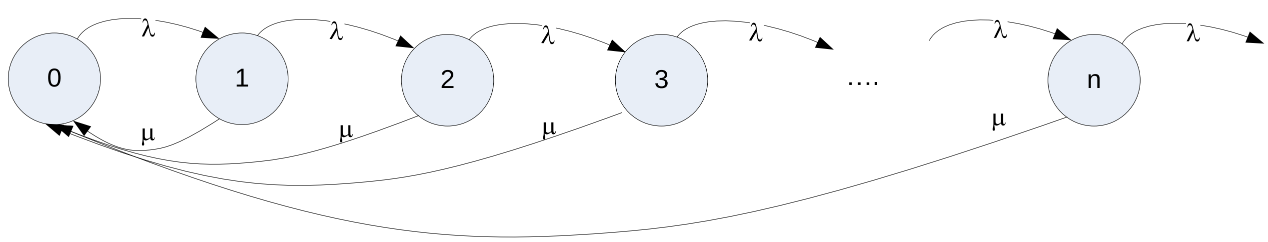
3) The result is given by the CLT: RV $Y = \sum_{j=1}^{100} X_j$ is approximately Normal, with a null mean and a variance 200 (for both cases 1) and 2), since the CLT requires only that the variance is finite). Therefore, it is: $P\{Y > 10\} = P\left\{\frac{Y-0}{\sqrt{200}} > \frac{10-0}{10\sqrt{2}}\right\} = P\left\{Z > \frac{1}{\sqrt{2}}\right\} = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right) \simeq 0.239$.

4) It is $P\{X_1 + X_2 = 0\} = P\{X_1 = a, X_2 = -a\} + P\{X_1 = -a, X_2 = +a\} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$. Moreover, the sum of $2n$ RVs is null if half of them are equal to $-a$, and the other half to $+a$. Each combination of "half and half" has a probability $\frac{1}{2^{2n}}$, and there are $\binom{2n}{n}$ such combinations, so the result is $P\{\sum_{j=1}^{2n} X_j = 0\} = \binom{2n}{n} \frac{1}{2^{2n}}$.

5) RV $S_n = \sum_{j=1}^{2n} X_j$ can take on values $-2n \cdot a$, $-2(n-1) \cdot a$, …, 0, …, $+2(n-1) \cdot a$, $+2n \cdot a$, i.e. $\{2a \cdot j,\ -n \le j \le +n\}$, and its PMF is obviously symmetric. Each of the above values $2a \cdot j$ can be obtained by selecting $n - j$ "positives" and $n + j$ "negatives", which can happen in $\binom{2n}{n-j} = \binom{2n}{n+j}$ different ways, each with probability $\frac{1}{2^{2n}}$. Therefore, the PMF is $p_n(2 \cdot a \cdot j) = \binom{2n}{n-j} \cdot \frac{1}{2^{2n}}$, $-n \le j \le +n$.

Exercise 2 – solution

The CTMC is the following:



Where each state represents the number of requests waiting.

2) The global equilibrium equations are:

$$\begin{cases} p_0 \cdot \lambda = \sum_{i=1}^{+\infty} p_i \cdot \mu \\ p_j \cdot (\lambda + \mu) = p_{j-1} \cdot \lambda \quad j \geq 1 \end{cases}$$

From the second one we easily obtain $p_j = p_0 \cdot \frac{\lambda^j}{(\lambda+\mu)^j}$.

From the above, normalization reads: $p_0 \cdot \left[1 + \sum_{j=1}^{+\infty} \frac{\lambda^j}{(\lambda+\mu)^j}\right] = 1$.

3) Call $\theta = \frac{\lambda}{\lambda+\mu}$, If $\mu \neq 0$ it is $\theta < 1$, hence the above sum converges to $\sum_{j=1}^{+\infty} \theta^j = \frac{\theta}{1-\theta} = \frac{\frac{\lambda}{\lambda+\mu}}{1-\frac{\lambda}{\lambda+\mu}} = \frac{\lambda}{\mu}$.
Therefore, the stability condition is $\mu \neq 0$.

We have $p_0 = \frac{\mu}{\lambda+\mu}$, $p_j = p_0 \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$. Hence, $p_j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$, $j \geq 0$.

Therefore, we have $P(z) = \sum_{j=0}^{+\infty} p_j \cdot z^j = \frac{\mu}{\lambda+\mu} \cdot \sum_{j=0}^{+\infty} \left(\frac{\lambda \cdot z}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda(1-z)+\mu}$. From the latter, we get:

$P'(z) = \frac{\partial}{\partial z} \frac{\mu}{\lambda(1-z)+\mu} = \frac{\lambda \cdot \mu}{[\lambda(1-z)+\mu]^2}$, and it is: $E[N] = P'(1) = \frac{\lambda}{\mu}$.

4) The mean response time is the mean service time (all requests are served simultaneously), i.e. $E[R] = \frac{1}{\mu}$.

5) The throughput is $\mu \cdot E[N] = \mu \cdot \frac{\lambda}{\mu} = \lambda$ (which is obvious, given that the system is stable).