PECSN 26/7/23

Exercise 1

Consider an *unfair* die, where the probability of obtaining 6 is $p \neq \frac{1}{6}$. The die is thrown several times. Call T the RV that counts the number of throws before a 6 appears for the first time, and assume that $E[T] = 3$.

1) Find the distribution of T and compute $P\left\{-\frac{1}{3} \leq T \leq \sqrt{11}\right\}$.
2) Let T_1, T_2 be two IID RVs having the same distribution as T. Find the PMF of $Z = min(T_1, T_2)$. Explain your findings.
3) Compute the 95th percentile of Z and $Var\left(\frac{25}{9}Z - \sqrt{6}\right)$.

Exercise 2

Consider an operating system where N tasks may issue a blocking request to a server. When *all* the tasks are blocked, the server unblocks *a random number of them simultaneously,* which then resume their operation. The other tasks remain blocked until the next service epoch. Each unblocked task issues blocking requests at a rate λ, and the service rate of the server is equal to μ. Call π_n the probability of unblocking n tasks.

1) Draw the CTMC
2) Compute the stability condition and the steady-state probabilities
   Assuming from now on that $\pi_n = const$,
3) Specialize the SS probabilities
4) Find the condition under which the PMF of the SS probabilities is a monotonic sequence.
5) Compute the SS probabilities seen by a blocking task.

Exercise 1 – solution

1) Quite obviously T is a geometric RV. Since $E(T) = \frac{1-p}{p} = 3$. Therefore, it is $p = \frac{1}{4}$. For a geometric RV, we have $F_T(k) = 1-(1-p)^{k+1}$. This means that $P\left\{-\frac{1}{3} \le T \le \sqrt{11}\right\} = \sum_{j=0}^{3} P\{T = j\} = F_T(3) = 1 - (1-p)^4 = \frac{175}{256}$.

2) We have:

$$
\begin{aligned}
P\{Z > k\} &= P\{T_1 > k, T_2 > k\} \\
&= P\{T_1 > k\} \cdot P\{T_2 > k\} \\
&= \left(1 - F_T(k)\right)^2 \\
&= (1-p)^{2\cdot(k+1)}
\end{aligned}
$$

Therefore, it is $F_Z(k) = 1-(1-p)^{2\cdot(k+1)} = F_T(2k)$. From the latter, we obtain:

$$
\begin{aligned}
p_Z(k) &= F_Z(k) - F_Z(k-1) \\
&= \left[1-(1-p)^{2\cdot(k+1)}\right] - \left[1-(1-p)^{2k}\right] \\
&= (1-p)^{2k}\left[1-(1-p)^{2\cdot}\right]
\end{aligned}
$$

Call $q = (1-p)^2$, and the latter becomes $p_Z(k) = q^k(1-q)$, which is again a geometric RV, with a success probability equal to $1-q = 1-(1-p)^{2\cdot}$.

Like with exponential RVs (of which the geometric are the discrete counterparts), a *minimum* property can be formulated: the min of 2 IID geometric RVs is itself a geometric RV, whose success probability is $1 - (1-p)^{2\cdot}$, i.e. the complement of the probability that *both* trials will fail.

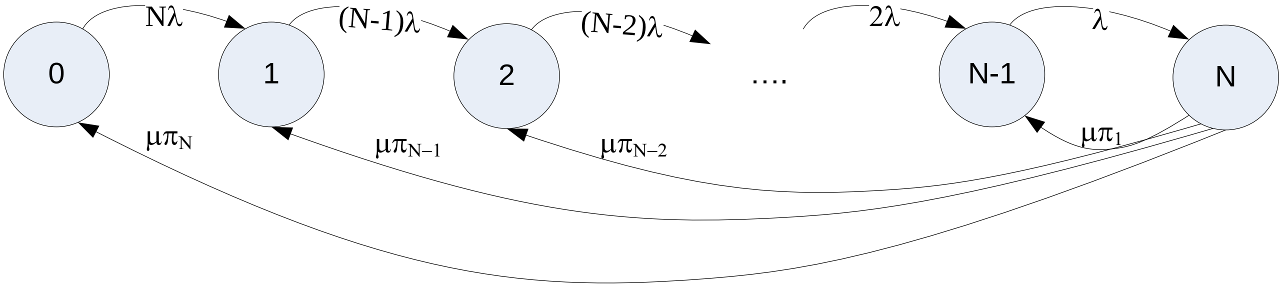
3) The 95th percentile of *Z* is obtained by solving the following equation $F_Z(k) = 0.95$ for *k*:

$$
\begin{aligned}
F_Z(k) &= 0.95 \\
1-(1-p)^{2\cdot(k+1)} &= 0.95 \\
\left(\frac{3}{4}\right)^{2\cdot(k+1)} &= \frac{1}{20} \\
2(k+1)\cdot\left[\log(3) - \log(4)\right] &= -\log 20 \\
k &= \left\lceil \frac{1+\log(2)}{2\left[\log(4)-\log(3)\right]} \right\rceil - 1 = 5
\end{aligned}
$$

Moreover, it is:

$$
\begin{aligned}
Var\left(\frac{25}{9}Z - \sqrt{6}\right) &= \left(\frac{25}{9}\right)^2 Var(Z) \\
&= \left(\frac{25}{9}\right)^2 \cdot \frac{1-\left[1-q^{2\cdot}\right]}{\left[1-q^{2\cdot}\right]^2}, \\
&= \frac{25^2}{81} \cdot \frac{81}{256} \cdot \frac{256^2}{25^2 \cdot 7^2} \\
&= \frac{256}{49}
\end{aligned}
$$

Exercise 2 - Solution



Note that the system has a finite capacity, hence it is always stable.

2) The steady-state global equations are the following:

$P_0 \cdot N \cdot \lambda = P_N \cdot \pi_N \cdot \mu,$

$P_j \cdot (N-j) \cdot \lambda = P_{j-1} \cdot (N-(j-1)) \cdot \lambda + P_N \cdot \pi_{N-j} \cdot \mu, \ \ 1 \le j \le \mathrm{N}-1$

$P_N \cdot \mu = P_{N-1} \cdot \lambda.$

After a few algebraic manipulations, the following recurrence can be easily obtained:

$$P_j = P_0 \cdot \frac{N}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N}$$

With $0 \le j \le \mathrm{N}-1$, and

$$P_N = P_0 \cdot \frac{N \cdot \lambda}{\mu \cdot \pi_N}$$

From the above we obtain the following normalization condition:

$$P_0 \cdot \left(\frac{N \cdot \lambda}{\mu \cdot \pi_N} + \sum_{j=0}^{N-1} \left[\frac{N}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N} \right] \right) = 1$$

From which we obtain the SS probabilities:

$$P_j = \frac{\frac{1}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N}}{\frac{\lambda}{\mu \cdot \pi_N} + \sum_{k=0}^{N-1} \left[\frac{1}{N-k} \cdot \sum_{i=0}^{k} \frac{\pi_{N-i}}{\pi_N} \right]}$$

$$P_N = \frac{\frac{\lambda}{\mu \cdot \pi_N}}{\frac{\lambda}{\mu \cdot \pi_N} + \sum_{k=0}^{N-1} \left[\frac{1}{N-k} \cdot \sum_{i=0}^{k} \frac{\pi_{N-i}}{\pi_N} \right]}$$

3) If $\pi_n = const$ it is $\pi_n = 1/N$, hence the above SS probabilities become:

$$P_j = \frac{\frac{j+1}{N-j}}{\frac{N\lambda}{\mu}+\sum_{k=0}^{N-1}\frac{k+1}{N-k}}, P_N = \frac{\frac{\lambda}{\mu\cdot\pi_N}}{\frac{N\lambda}{\mu}+\sum_{k=0}^{N-1}\frac{k+1}{N-k}}$$

Considering that $\sum_{k=0}^{N-1}\frac{k+1}{N-k} = \sum_{n=1}^{N}\frac{N-n+1}{n} = \sum_{n=1}^{N}\left(\frac{N+1}{n}-1\right) = (N+1)H_N - N$, we obtain:

$$P_j = \frac{\frac{j+1}{N-j}}{(N+1)H_N+N\left(\frac{\lambda}{\mu}-1\right)}, P_N = \frac{\frac{N\lambda}{\mu}}{(N+1)H_N+N\left(\frac{\lambda}{\mu}-1\right)}$$

Where H_N is the *N-th* harmonic number.

4) Let us check that $P_j < P_{j+1}$. In fact, $\frac{P_{j+1}}{P_j} = \frac{j+2}{N-j-1-}\frac{N-j}{j+1} > 1$. Moreover, it is $P_{N-1} < P_N$ if

$$\frac{(N-1)+1}{N-(N-1)} < \frac{N\lambda}{\mu}$$

i.e., if $\lambda > \mu$. Therefore, under the above condition the PMF of the SS probabilities is an increasing sequence.

5) The system is non PASTA, hence:

$$r_j = \frac{\lambda_j P_j}{\sum_{i=0}^{N}\lambda_i P_i} = \frac{(N-j)P_j}{\sum_{i=0}^{N-1}(N-i)P_i} = \frac{j+1}{\sum_{i=1}^{N} i} = 2\frac{j+1}{N\cdot(N+1)}$$

With $0 \le j \le N-1$.