PECSN 5/7/2023

Exercise 1

Consider the following function, where $\lambda > 0$:

$$f(t) = A_\lambda \cdot \begin{cases} e^{-t^2/2}, & t < 0 \\ e^{-\lambda t}, & t \geq 0 \end{cases}$$

1) Draw $f(t)$ with the maximum possible accuracy;
2) Compute constant A_λ (as a function of λ) so that the above function is a PDF for RV X;
3) Compute the expectation of X and find the values of λ for which $E[X]$ is, respectively, negative, null and positive;
4) Compute the CDF and the PDF of $|X|$

Exercise 2

Consider a manufacturing plant where there are two processing stations (PS). Jobs arrive at PS 1, they are put on a conveyor belt, and they are processed FIFO by two identical machines, which may work in parallel. After leaving PS1, jobs get to PS2. Before arriving at PS2, they are inspected, and sent back to PS1 if they are found not compliant. This happens with probability π_1. Jobs entering PS2 are processed by a single machine. After leaving PS2, jobs are either completed or they can be sent back to PS1 again, this time with a probability π_2.

Interarrivals at the plant are exponentially distributed, and so are the service times of single machines at PS1 and PS2. Call γ the arrival rate of jobs, μ_1, μ_2 the serving rates of the machines at PS1, PS2.

1) Model the above plant as a queueing network. Compute the routing matrix and the arrival rates. State explicitly the conditions under which the computations are correct.

2) Compute the conditions under which PS1 (PS2) is the bottleneck. Verify your answer in limit cases and write down an intuitive justification.

3) Compute an upper bound on the arrival rate for the system to achieve stability, and the steady-state probability to have *n* jobs in PS2 *conditioned to the fact that there are k jobs in PS1*.

4) Compute the average number of visits to PS1 and PS2. State the conditions under which

a) PS *j* has less than one visit on average

b) PS1 is visited more often than PS2

Exercise 1 – solution

1) Function $f(t)$ has a standard normal shape on the left semi-axis, and an exponentially decaying one in the positive semi-axis. It is continuous in 0, where its value is equal to A.
2) In order for $f(t)$ to be a PDF, normalization must hold. The integral can be split into two, and the following observations are in order:
   - $\int_0^{+\infty} A_\lambda \cdot e^{-\lambda t} dt = A_\lambda / \lambda$
   - $\int_{-\infty}^{0} A_\lambda \cdot e^{-t^2/2} dx = \frac{A_\lambda}{2}\sqrt{2\pi}$

The last observation is due to the fact that a standard normal is symmetric. Therefore, normalization reads $A_\lambda \cdot (\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}) = 1$, i.e.,

$$A_\lambda = \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}}$$

3)

$$E[X] = \int_{-\infty}^{+\infty} t \cdot f(t) dt$$

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(\int_{-\infty}^{0} t \cdot e^{-t^2/2} dt + \int_0^{+\infty} t \cdot e^{-\lambda t} dt \right)$$

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(-\int_{-\infty}^{0} -t \cdot e^{-t^2/2} dt + \frac{1}{\lambda}\int_0^{+\infty} t \cdot \lambda \cdot e^{-\lambda t} dt \right)$$

The first integral can be written in the form $f'(x)e^{f(x)}$, hence a primitive is $e^{f(x)}$, whereas the second integral can be obtained from the mean value of an exponential.

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(-1 + \frac{1}{\lambda^2} \right) = \frac{1 - \lambda^2}{\lambda + \frac{\lambda^2}{2}\sqrt{2\pi}}$$

We easily obtain that E[X] is positive when $0 < \lambda < 1$, null when $\lambda = 1$ and negative when $\lambda > 1$.

4) Call $Y = |X|$.We easily obtain

$$F_Y(k) = P\{|X| \le k\} = P\{-k \le X \le k\} = \int_{-k}^{k} f(t) dt = \int_{-k}^{0} f(t) dt + \int_0^{k} f(t) dt$$

$$= A_\lambda \cdot \sqrt{2\pi} \int_{-k}^{0} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt + \frac{A_\lambda}{\lambda} \cdot \int_0^{k} \lambda \cdot e^{-\lambda t} dt$$

$$= A_\lambda \cdot \sqrt{2\pi} \left(\Phi(k) - \frac{1}{2} \right) + \frac{A_\lambda}{\lambda} \cdot \left(1 - e^{-\lambda k} \right)$$

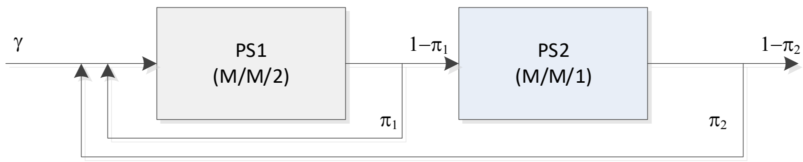
By derivation,

$$f_Y(k) = A_\lambda \cdot \left(e^{-k^2/2} + e^{-\lambda k} \right)$$

And the latter is just the sum of the two branches of X's PDF, which makes sense since the support is halved and the underlying area must still be one.

Exercise 2 - Solution

1) The system can be modeled as a queueing network as follows:



With the following routing matrix and arrival vector

$$\mathbf{\Pi} = \begin{bmatrix} \pi_1 & 1-\pi_1 \\ \pi_2 & 0 \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$$

From the figure it is clear that:

$$\begin{cases} \lambda_2 = \lambda_1 \cdot (1-\pi_1) \\ \lambda_1 = \gamma + \pi_1 \cdot \lambda_1 + \pi_2 \cdot \lambda_2 \end{cases}$$

From which one obtains

$$\begin{cases} \lambda_1 = \dfrac{\gamma}{(1-\pi_1)\cdot(1-\pi_2)} \\ \lambda_2 = \dfrac{\gamma}{1-\pi_2} \end{cases}$$

The above computations are correct if $\pi_j < 1$. In fact, condition $\pi_j = 1$ implies that no job come out of the plant, which means that the system cannot reach a steady state.

2) The bottleneck is the system with the highest utilization. The latter is:

$$\begin{cases} \rho_1 = \dfrac{\lambda_1}{2\mu_1} = \dfrac{\gamma}{2\mu_1\cdot(1-\pi_1)\cdot(1-\pi_2)} \\ \rho_2 = \dfrac{\lambda_2}{\mu_2} = \dfrac{\gamma}{\mu_2\cdot(1-\pi_2)} \end{cases}$$

Therefore, $\rho_1 > \rho_2$ implies

$$\frac{\gamma}{2\mu_1\cdot(1-\pi_1)\cdot(1-\pi_2)} > \frac{\gamma}{\mu_2\cdot(1-\pi_2)}$$

$$2\mu_1\cdot(1-\pi_1) < \mu_2$$

$$\frac{\mu_1}{\mu_2} < \frac{1}{2\cdot(1-\pi_1)}$$

If $\pi_1 = 0$, then PS1 and PS2 have exactly the same arrivals (they are in a tandem). Therefore, in this case, the server speed at PS1 needs be at least one half of PS2's, since PS1 has two servers. If, instead, $\pi_1 > 0$, PS1 needs to be faster than that, since it will also see arrivals that PS2 does not see, due to the first feedback loop.

3) The stability condition is that both SCs are stable, i.e., $\rho_1 < 1, \rho_2 < 1.$, i.e.

$$\gamma < \min\{2\mu_1\cdot(1-\pi_1)\cdot(1-\pi_2), \mu_2\cdot(1-\pi_2)\}$$

Under the above, the SS probabilities related to PS2 are independent of those of PS1, since OJN admit product forms. Hence conditioning makes no matter. The SS probabilities are those of an M/M/1 system, hence

$$p_n = \left(\frac{\gamma}{\mu_2 \cdot (1-\pi_2)}\right)^n \cdot \left(1 - \frac{\gamma}{\mu_2 \cdot (1-\pi_2)}\right)$$

4) The mean number of visits to each PS is the arrival rate scaled by the total rate of external arrivals, i.e., γ. Thus we have:

$$\begin{cases} v_1 = \dfrac{1}{(1-\pi_1)\cdot(1-\pi_2)} \\ v_2 = \dfrac{1}{1-\pi_2} \end{cases}$$

Condition a) is plainly impossible, since a job *must* traverse both PSs at least once in order to leave. Trying to impose it on the above equalities leads to the routing probabilities taking negative values. Condition b) is instead true if $\pi_1 > 0$.