Exercise 1

Consider the following function:

$$f(x) = \begin{cases} 0 & x < 1 \\ \alpha \cdot x^{-(\alpha+1)} & x \geq 1 \end{cases}$$

With α being an integer.

1) State the conditions under which $f(x)$ is a PDF;
2) Compute the CDF and the mean.

Let X be a RV distributed according to the above PDF.

3) Compute the PDF for RV $Y = log(X)$;
4) Assume that RVs $X_1, X_2, \dots, X_n$ are IID according to $F_X(x)$ computed at point 2. Compute the CDF for $W = \max_i\{1/X_i\}$. Discuss the behavior of the CDF as a function of n.

Exercise 2

SpeedyMatch is a blind speed-dating agency that arranges dates between strangers of opposite sexes. A male that wants to use their services shows up at their premises, and:

- If there are already females at the premises, he picks up the longest-waiting one and they leave (instantly);
- Otherwise, he queues up (FIFO) behind other males.

Dually, a female that wants to use SpeedyMatch shows up at their premises, and:

- If there are already males at the premises, she picks up the longest-waiting one and they leave (instantly);
- Otherwise, she queues up (FIFO) behind other females.

Speedymatch has K seats available for queueing at its premises, and it does not let customers queue up if they cannot seat. Males and females arrive with exponential interarrival times, at a rate λ_M and λ_F respectively.

1) Model the system as a queueing system and draw a CTMC;
2) Write down the local equilibrium equations and find the stability condition and the SS probabilities;
3) Find the probability that a customer is rejected;
4) Find the mean response time for a female customer (assuming λ_M = λ_F). Verify your answer in the limit case $K = 1$ and explain the result.

Exercise 1 - Solution

f(x) is a PDF if it is always non-negative and normalization holds. The first condition is verified if $\alpha \geq 0$. The second is verified if:

$$\int_1^{+\infty} \alpha \cdot x^{-(\alpha+1)} dx = 1$$

The above integral converges if and only if $\alpha \geq 1$, which is therefore the required condition. Under the latter hypothesis, we have:

$$F(x) = \int_1^{x} \alpha \cdot y^{-(\alpha+1)} dx \, [-y^{-\alpha}]_1^x = 1 - \frac{1}{x^\alpha}$$

The mean value is:

$$E[x] = \int_1^{+\infty} x \cdot \alpha \cdot x^{-(\alpha+1)} dx = \int_1^{+\infty} \alpha \cdot x^{-\alpha} dx$$

The above integral is infinite unless $\alpha \geq 2$. Under the latter hypothesis, we have:

$$E[x] = \frac{\alpha}{\alpha - 1} \cdot \left[-x^{-(\alpha-1)}\right]_1^{+\infty} = \frac{\alpha}{\alpha - 1}$$

$$F_Y(y) = P\{Y \leq y\} = P\{\log(X) \leq y\} = P\{\mathrm{X} \leq e^y\} = 1 - \frac{1}{(e^y)^\alpha} = 1 - e^{-\alpha y}$$

And, since $X \geq 1$, then $Y \geq 0$. Therefore, it is $f_Y(y) = \alpha \cdot e^{-\alpha y}$.

Call $Z = 1/X_i$. We have:

$F_Z(z) = P\{Z \leq z\} = P\{\mathrm{X} > 1/z\} = 1 - F_X(1/z) = z^\alpha$.

And, since $X \geq 1$, then $0 \leq Z \leq 1$.

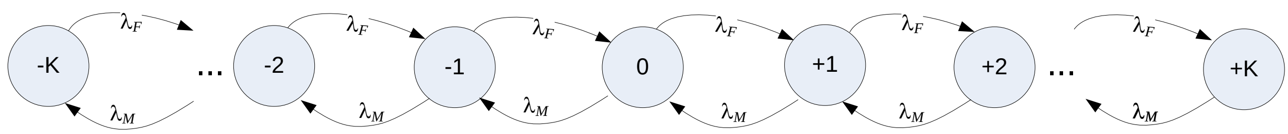
Now, it is:

$$F_W(w) = P\{Z_1 \leq w, Z_2 \leq w, \dots, Z_n \leq w\} = [F_Z(w)]^n = w^{n\alpha}$$

Since the support of RV Z is limited, then the CDF will decrease with n, and will tend to a step function as n goes to infinity.

Exercise 2 - Solution

1) Given the requirements, it is impossible that males and females queue up simultaneously. Therefore, if there is a queue, it will *only* contain either males or females, and it will be of at most K customers. When there is a queue, the arrival of a customer of the same sex increases the queue, and the arrival of a customer of the opposite sex decreases the queue. Thus, a state description for this system is given by the length of the queue and a binary information (male, female), which can be represented by a *sign* (arbitrarily, + is female and – is male). The system has $2K + 1$ states, and the transition diagram is the following:



The system is always stable since it has a finite number of states.

2) The local equilibrium equations are $p_j \cdot \lambda_F = p_{j+1} \cdot \lambda_M, \; -K \le j < K.$

From the above, one obtains straightforwardly that $p_j = p_{-K} \cdot \rho^{j+K}$, $-K \le j \le K$, where $\rho = \frac{\lambda_F}{\lambda_M}$.

The normalization condition reads $\sum_{-K}^{+K} p_j = 1$, hence:

$$\sum_{-K}^{K} p_{-K} \cdot \rho^{j+K} = p_{-K} \cdot \sum_{0}^{2K} \rho^j = \begin{cases} p_{-K} \cdot \dfrac{1-\rho^{2K+1}}{1-\rho} & if\ \rho \ne 1 \\ p_{-K} \cdot (2K+1) & if\ \rho = 1 \end{cases}$$

Therefore,

$$p_j = \begin{cases} \dfrac{1-\rho}{1-\rho^{2K+1}} \cdot \rho^{j+K} & if \rho \ne 1 \\ \dfrac{1}{2K+1} & if \rho = 1 \end{cases}, \qquad -K \le j \le K,$$

3) The probability that a customer is rejected is $p_R = p_{-K} + p_{+K}$. Therefore:

$$p_R = \begin{cases} \dfrac{1-\rho}{1-\rho^{2K+1}} (1+\rho^{2K}) & \rho \ne 1 \\ \dfrac{2}{2K+1} & \rho = 1 \end{cases}$$

4) This is a non-PASTA system. The mean arrival rates for females and males are, respectively:

$$\overline{\lambda_F} = \lambda_F \cdot (1 - p_K)$$

$$\overline{\lambda_M} = \lambda_M \cdot (1 - p_{-K})$$

The response time of a female client is:

- null, if males are queued, i.e., with probability $p = \sum_{-K}^{-1} r_j$, where

$$r_j = \frac{\lambda_F}{\overline{\lambda_F}} \cdot p_j = \frac{p_j}{1-p_K} = \frac{1}{2K}$$

- equal to the sum of the arrival times of $j+1$ males if she arrives when the system is in state j (i.e., with probability r_j, $0 \le j \le K-1$). The sum of $j+1$ exponentials is an Erlang distribution with as many stages, and its mean value is $(j+1)/\lambda_M$. Therefore:

$$E[R] = \sum_{j=0}^{K-1} \frac{j+1}{\lambda_M} \cdot \frac{1}{2K} + \sum_{j=-K}^{-1} 0 \cdot \frac{1}{2K} = \frac{1}{\lambda_M} \cdot \frac{1}{2K} \cdot \frac{K \cdot (K+1)}{2} = \frac{1}{\lambda_M} \cdot \frac{K+1}{4}$$

In the limit case $K = 1$ the above expression boils down to $\frac{1}{\lambda_M} \cdot \frac{1}{2}$. In fact, in this case an arrival of a female may occur only in state -1 or 0, with equal probability. In the first case, the response time will be zero, and in the second case it will be the mean arrival time of a male customer, which explains the result.