PECSN, 22/02/23

Exercise 1
Consider the following JPDF:

$$f(x,y) = \begin{cases} k \cdot x \cdot y & 0 \le x \le 2, \quad 0 \le y \le x \\ 0 & otherwise \end{cases}$$

1) Find k
2) Compute the PDFs of RVs X and Y. Are X and Y independent RVs? Justify your answer.
3) Compute the PDF of RV $Z = \frac{1}{X}$ and its mean value. Is Z heavy-tailed? Justify your answer.

Exercise 2
Consider a queueing system where jobs arrive with exponential interarrival times. The arrival rate is independent of the number of jobs currently in the system. The service time is also exponential, and the service rates may depend on the number of jobs in the system (and also be null in some states). Whenever a job ends its service, *all the waiting jobs are flushed*, i.e., the queue is emptied.

1) Model the system as a queueing system and draw the CTMC.
2) Write down the global equilibrium equations and the normalization condition in their general form.

Assume that service rates are *constant*:

3) Compute the stability condition.
4) Compute the steady-state probabilities and the mean and variance of the number of jobs in the system.
5) What happens to stability if $\mu_j = 0 \; \forall j \ge k$? Justify your answer.

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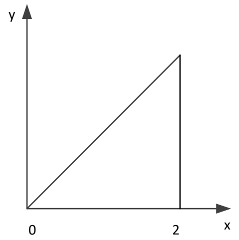
Exercise 1 – solution

The JPDF is defined in the triangle in the figure.



1) The normalization condition reads $\int_0^2 \left[\int_0^x k \cdot x \cdot y \ \ dy\right] \ dx = 1$, hence:

$$\int_0^2 \left[\int_0^x k \cdot x \cdot y \ \ dy\right] \ dx = k \cdot \int_0^2 x \cdot \left[\int_0^x y \ \ dy\right] \ dx = \ k \cdot \int_0^2 x \cdot \frac{x^2}{2} \ dx = k \cdot \left[\frac{x^4}{8}\right]_0^2 = 2k$$

The result is $k = \frac{1}{2}$.

The same result can be obtained inverting the order of the variables in the double integral, taking care to define the extremes correctly:

$$\int_0^2 \left[\int_y^2 k \cdot x \cdot y \ \ dx\right] \ dy = \cdots = 2k$$

2) The PDFs are:

$$f_X(x) = \frac{1}{2} x \cdot \int_0^x y \ \ dy = \frac{1}{4} x^3$$

$$f_Y(y) = \frac{1}{2} y \cdot \int_y^2 x \ \ dx = \frac{1}{2} y \cdot \left[2 - \frac{y^2}{2}\right] = y - \frac{y^3}{4}$$

It can easily be checked that $f(x, y) \neq f_X(x) \cdot f_Y(y)$, hence the two RVs are not independent.

3) $P\{X \leq x\} = F_X(x) = \frac{x^4}{16} = P\left\{\frac{1}{X} \geq \frac{1}{x}\right\} = 1 - F_Z\left(\frac{1}{x}\right)$. Therefore, $F_Z(z) = 1 - \frac{1}{16 \cdot z^4}$, $z \geq \frac{1}{2}$.

The PDF is $f_Z(z) = \frac{1}{4z^5}$. The mean value of Z is

$$E[Z] = \int_{\frac{1}{2}}^{+\infty} \frac{1}{4z^5} \cdot z \ \ dz = \frac{1}{4} \int_{\frac{1}{2}}^{+\infty} \frac{1}{z^4} \ \ dz = \frac{1}{4}\left[-\frac{1}{3z^3}\right]_{\frac{1}{2}}^{+\infty} = \frac{2}{3}$$

The definition of heavy-tail is the following: $\forall \lambda > 0, \quad \lim_{z \to \infty} e^{\lambda z} \cdot \left(1 - F_Z(z)\right) = \infty$

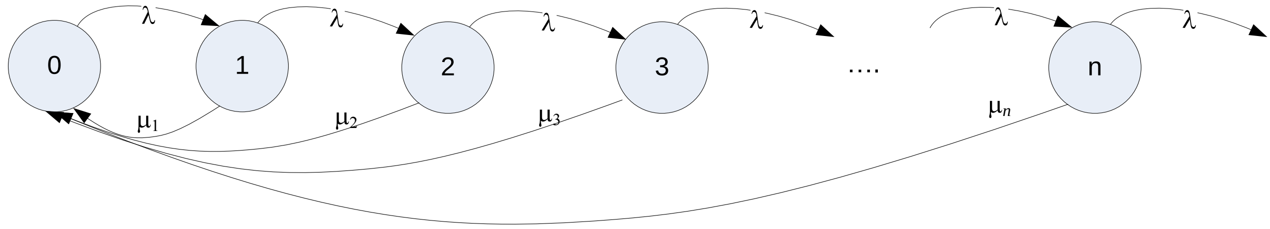
By substitution, we get:

$$\lim_{z \to \infty} e^{\lambda z} \cdot \frac{1}{16 z^4} = +\infty, \forall \lambda > 0.$$

Therefore, Z is heavy-tailed.

Exercise 2 – solution

1) The CTMC is the following:



2) The global equilibrium equations in their general form are:

$$\begin{cases} p_0 \cdot \lambda = \sum_{i=1}^{+\infty} p_i \cdot \mu_i \\ p_j \cdot (\lambda + \mu_j) = p_{j-1} \cdot \lambda \quad j \geq 1 \end{cases}$$

From the second one we easily obtain $p_j = p_0 \cdot \frac{\lambda^j}{\prod_{i=1}^{j}(\lambda+\mu_i)}$.

From the above, normalization reads: $p_0 \cdot \left[1 + \sum_{j=1}^{+\infty} \frac{\lambda^j}{\prod_{i=1}^{j}(\lambda+\mu_i)}\right] = 1$.

3) If service rates are state-independent, it is $\mu_j = \mu$, hence the infinite sum in the normalization becomes: $\sum_{j=1}^{+\infty} \frac{\lambda^j}{(\lambda+\mu)^j}$

Call $\theta = \frac{\lambda}{\lambda+\mu}$, If $\mu \neq 0$ it is $\theta < 1$, hence the above sum converges to $\sum_{j=1}^{+\infty} \theta^j = \frac{\theta}{1-\theta} = \frac{\frac{\lambda}{\lambda+\mu}}{1-\frac{\lambda}{\lambda+\mu}} = \frac{\lambda}{\mu}$.

Therefore, the stability condition is $\mu \neq 0$.

4) With non-null, state-independent service rates, it is $p_0 = \frac{\mu}{\lambda+\mu}$, $p_j = p_0 \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$.

Hence, $p_j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}, j \geq 0$.

Therefore, we have $P(z) = \sum_{j=0}^{+\infty} p_j \cdot z^j = \frac{\mu}{\lambda+\mu} \cdot \sum_{j=0}^{+\infty} \left(\frac{\lambda \cdot z}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda(1-z)+\mu}$. From the latter, we get:

$$P'(z) = \frac{\partial}{\partial z} \frac{\mu}{\lambda(1-z)+\mu} = \frac{\lambda \cdot \mu}{[\lambda(1-z)+\mu]^2}$$

$$P''(z) = \frac{\partial}{\partial z} P'(z) = \frac{\lambda \cdot \mu \cdot [2\lambda^2 z - 2\lambda(\lambda+\mu)]}{[\lambda(1-z)+\mu]^4}$$

And it is:

$$E[N] = P'(1) = \frac{\lambda}{\mu},$$

$$Var(N) = P"(1) + P'(1) - P'(1)^2 = \frac{2\lambda^2}{\mu^2} + \frac{\lambda}{\mu} - \frac{\lambda^2}{\mu^2} = \frac{\lambda}{\mu} \cdot \left(1 + \frac{\lambda}{\mu}\right)$$

5) If the service rates are identically null starting from some state k onward, then it is $p_j = p_k \ \forall j \geq k$. Then, normalization reads:

$$p_0 + \sum_{j=1}^{k-1} p_j + \sum_{j=k}^{+\infty} p_k = 1$$

The infinite sum diverges, unless $p_k = 0$. This means that the system is unstable. This could have been observed right from the CTMC: if the system hits state k, which happens with nonnull probability, then it will never empty again, because there is no longer a path leading to state zero.