Exercise 1

Consider the following JPDF of two RVs X and Y:

$$f(x,y) = \begin{cases} K & 0 \leq x \leq 1, \quad 0 < y < 1 - x \\ 0 & otherwise \end{cases}$$

where K is a real-valued constant.

1) Compute K *without using integration*;
2) Compute $P\{Y > \alpha\}$,$0 < \alpha < 1$, *without using integration*;
3) Compute $f_X(x), f_Y(y)$, and state if the two RVs are independent;
4) Compute the PDF of RV $W = \frac{Y}{X}$.

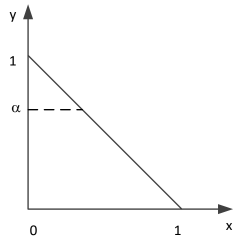
Exercise 2

Consider a system where a constant (and configurable) number of jobs K is processed cyclically through *three stages*. Stage j, $0 \leq j \leq 2$, is populated by $3 - j$ *identical* Service Centers, each one having a single server with a serving rate μ_j. Jobs leaving a SC in stage j go to stage $|j + 1|_3$, and they are routed to each of the SCs of the new stage with the same probability.

1) Model the system as a closed queueing network
2) Find the routing matrix and compute the SS probabilities in their general form.
3) Explain the SS probabilities and find an alternative formulation that only includes the number of jobs at each *stage*.
4) Find $G(M,K)$ for a generic K assuming $\mu_1 = \frac{3}{2}\mu_0$, $\mu_2 = 3\mu_0$. Instantiate the result for $K = 9$.
5) Find the probability that all the jobs are at stage j

Exercise 1 – Solution

1) As per the figure, the JPDF is non null and equal to K within the triangle. The integral of that constant in a triangular area must be equal to 1. Therefore, it must be that $K = 2$. The same result can be obtained (more tediously) through integration.



2) With reference to the above drawing, $\{Y > \alpha\}$ is the triangle having the $(0,1)$,$(0,\alpha)$,$(1-\ \alpha,\alpha)$ as vertices. Since the JPDF is uniform, and its integral on the whole triangle is equal to one, the requested probability is just the proportion of the areas, i.e.

$$P\{Y > \alpha\} = \frac{\frac{(1-\alpha)^2}{2}}{\frac{1}{2}} = (1-\alpha)^2$$

3) By integration:

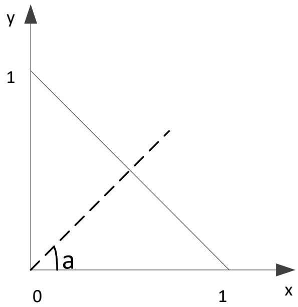
$$f_X(x) = \int_0^{1-x} f(x,y)dy = \int_0^{1-x} 2dy = 2(1-x),\ f_Y(y) = \int_0^{1-y} f(x,y)dx = \int_0^{1-y} 2dx = 2(1-y)$$

Thus, $f(x,y) \neq f_X(x) \cdot f_Y(y)$, hence the two RVs are not independent.

4) RV W is defined in $0,+\infty)$. It is $P\{W \leq a\} = P\{Y \leq a \cdot X\}$. The above inequality defines a triangle whose vertexes are (0,0), (1,0), $\left(\frac{1}{(1+a)},\frac{a}{(1+a)}\right)$, hence its base is equal to 1 and its height is equal to $\frac{a}{(1+a)}$. Following the same reasoning as for points 1 and 2, the probability is the area of that triangle normalized to the area of the initial triangle, i.e.

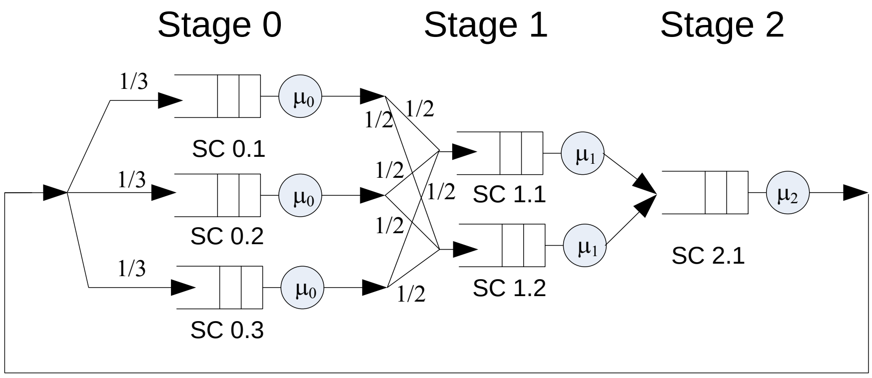
$$P\{W \leq a\} = P\{Y \leq a \cdot X\} = \frac{\frac{\left[\frac{a}{(1+a)}\right]}{2}}{\frac{1}{2}} = \frac{a}{(1+a)}$$

The PDF for W is the following: $f_W(a) = \frac{\partial}{\partial a}\left(\frac{a}{1+a}\right) = \frac{1}{(1+a)^2}$



Exercise 2 - solution

1) A model for the above system is the following:



The routing matrix is the following (row indexes are also reported for ease of reading):

$$\boldsymbol{\Pi} = \begin{matrix} 0.1 \\ 0.2 \\ 0.3 \\ 1.1 \\ 1.2 \\ 2.1 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}$$

Hence the routing equations are:

$$\begin{cases} \lambda_{0.x} = \frac{1}{3}\lambda_{2.1} \quad 1 \le x \le 3 \\ \lambda_{1.y} = \frac{3}{2}\lambda_{0.x} \quad 1 \le x \le 3, \quad 1 \le y \le 2 \\ \lambda_{2.1} = 2 \cdot \lambda_{1.y} \quad 1 \le y \le 2 \end{cases}$$

2) From the above system (also due to reasons of symmetry), it is clear that $\lambda_{0.x} = \lambda_0, \lambda_{1.y} = \lambda_1$. A solution of the above system is $\boldsymbol{e}^T = \left[e, \quad e, \quad e, \quad \frac{3}{2}e, \quad \frac{3}{2}e, \quad 3e\right]$, with *e* being an arbitrary constant. Therefore, we can select $e = \mu_0$ and obtain $\boldsymbol{\rho}^T = \left[1, \quad 1, \quad 1, \quad \frac{3\mu_0}{2\mu_1}, \quad \frac{3\mu_0}{2\mu_1}, \quad 3\frac{\mu_0}{\mu_2}\right]$. By Gordon and Newell's Theorem, the SS probabilities are:

$$p(n_{0.1}, n_{0.2}, \ldots, n_{2.1}) = \frac{1}{G(M,K)} \cdot \left(\frac{3\mu_0}{2\mu_1}\right)^{(n_{1.1}+n_{1.2})} \cdot \left(\frac{3\mu_0}{\mu_2}\right)^{n_{2.1}}$$

3) The above expression can be rewritten by observing that only the number of jobs *at each stage* matters, since probabilities are equal for every distribution of jobs within a stage:

$$p(K-(N_1+N_2), N_1, N_2) = \frac{1}{G(M,K)} \cdot \left(\frac{3\mu_0}{2\mu_1}\right)^{N_1} \cdot \left(\frac{3\mu_0}{\mu_2}\right)^{N_2}$$

4) Under the hypotheses, it is $\rho = 1$ at each stage. This means that $G(M,K) = |\mathcal{E}| = \binom{K+M-1}{M-1}$. With $K = 9$ we have $G(M,K) = \binom{14}{5} = 2002$.

5) The probability of each state is the same, and it is equal to

$$p = \frac{1}{G(M,K)} = \frac{1}{2002}$$

Therefore, we are in a UPM. The probability that all jobs are at stage j is $P_j = \frac{|E|}{|\mathcal{E}|}$, where $|E|$ is the number of possible combinations of K jobs at $3-j$ SCs, i.e., $|E| = \binom{K+(3-j)-1}{(3-j)-1}$. This yields:

$$P_0 = \frac{\binom{11}{2}}{2002} = \frac{55}{2002}$$

$$P_1 = \frac{\binom{10}{1}}{2002} = \frac{10}{2002}$$

$$P_2 = \frac{\binom{9}{0}}{2002} = \frac{1}{2002}$$