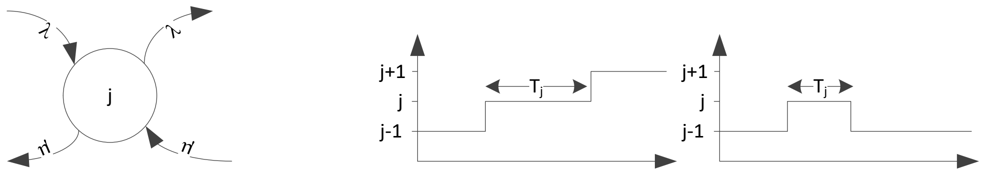
PECSN, 18 Jan 2023

Exercise 1

Consider a state $j > 0$ of a birth-death system like the one in the figure to the left, with exponential interarrival and service times, assumed to be independent. Let λ be the arrival rate and μ be the service rate, both constant.



Call T_j the RV that describes the <u>*time*</u> spent by the system in a <u>*single visit*</u> to state j, two samples of which are shown in the figure to the right. Do not confuse T_j with the proportion of time spent in state j at the steady state.

1) Compute the CDF of T_j;
2) Compute the mean value and variance of T_j;
3) Assume now that the system has *bulk* arrivals/services: the arrival rate is still λ and the service rate is still μ, but each arrival (service) can jump k states to the left (right) with some probability p_k (r_k). Answer again questions 1 and 2.

Exercise 2

Consider a system where service times are exponentially distributed with a mean $\frac{1}{\mu}$. The system is fed by *two independent and identical input processes*. Both inputs are Poisson processes with a rate λ. The first one is always active, and the second one is active only when an *odd* number of customers is in the system.

1) Model the system as a queueing system and draw the CTMC;
2) Write down the *local* equilibrium equations;
3) Derive the stability conditions and the steady-state probabilities;
4) Find the system throughput. Compare it to λ and justify the result;
5) Compute the steady-state probabilities r_j seen by an arriving customer.

PECSN, 18 Jan 2023

Exercise 1 - solution

1) Assume the system enters state j (say, at time t). It then leaves that state because either an arrival occurs, or a departure does. The time of either event is an exponential RV, with a rate λ and μ respectively. The fact that exponential RVs are memoryless implies that the *residual* interarrival/service times *seen at time t* are still exponential, and with the same rates.
Call T_A and T_B the RVs of an interarrival and service time. It is:

$$F_j(\tau) = P\{T_j \le \tau\} = P\{min(T_A, T_B) \le \tau\} = 1 - e^{-(\lambda+\mu)\cdot\tau}$$

Since the minimum among n independent exponential RVs is an exponential RV whose rate is the sum of the rates.

2) The answer is straightforward: $E[T_j] = \frac{1}{\lambda+\mu}$, $Var(T_j) = \frac{1}{(\lambda+\mu)^2}$.

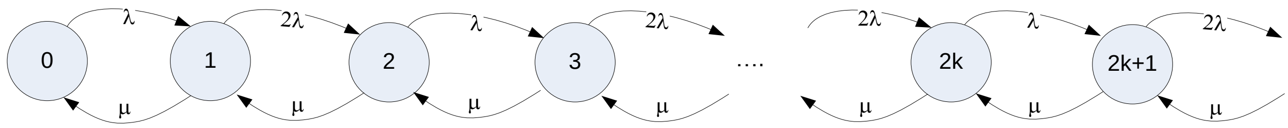
3) The answer is still the same, since:

a) the computations are unaffected by the probability of *entering* state j.

b) only the time at which arrivals and departures occur matters, whereas the length of the (right/left) arcs does not.

Exercise 2 - Solution

1) The CTMC is the following (note that the superimposition of two independent Poisson processes is a Poisson process):



2) In order to write down the equations, it pays to distinguish *odd* and *even* states. The local equilibrium equations are:

$$\begin{cases} \lambda \cdot p_{2k} = \mu \cdot p_{2k+1} & k \ge 0 \\ 2\lambda \cdot p_{2k-1} = \mu \cdot p_{2k} & k > 0 \end{cases}$$

from which we obtain – after some straightforward algebraic manipulations:

$$\begin{cases} p_{2k} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot p_0 & k \ge 0 \\ p_{2k+1} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot p_0 & k \ge 0 \end{cases}$$

3) The stability conditions is the following:

$$p_0 \cdot \left[\sum_{k=0}^{+\infty} 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} + \sum_{k=0}^{+\infty} 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1}\right] = 1$$

$$p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right) \cdot \sum_{k=0}^{+\infty} \left(\frac{\sqrt{2}\cdot\lambda}{\mu}\right)^{2k} = 1$$

$$p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right) \cdot \sum_{k=0}^{+\infty} \left[\left(\frac{\sqrt{2}\cdot\lambda}{\mu}\right)^{2}\right]^k = 1$$

The series converges if and only if $\sqrt{2}\cdot\lambda < \mu$. Under the above condition, we obtain:

$$p_0 \cdot \left(1+\frac{\lambda}{\mu}\right) \cdot \frac{1}{1-2\frac{\lambda^2}{\mu^2}} = 1$$

$$p_0 \cdot \frac{\lambda+\mu}{\cancel{\mu}} \cdot \frac{\mu^{\cancel{2}}}{\mu^2 - 2\lambda^2} = 1$$

$$p_0 = \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)}$$

Thus, by applying the above formulas:

$$\begin{cases} p_{2k} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)} & k \geq 0 \\ p_{2k+1} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)} & k \geq 0 \end{cases}$$

4) The system throughput can be found by applying the formula: $\gamma = \sum_{j=1}^{+\infty} \mu_j \cdot p_j = \mu \cdot (1 - p_0) = \mu \cdot \left[1 - \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)}\right] = \lambda \cdot \frac{2\lambda+\mu}{\lambda+\mu}$.

   It is $\gamma > \lambda$, which makes sense since there are states when the arrival rate is 2λ.

5) The system is non-PASTA, hence $r_j = \frac{\lambda_j}{\bar{\lambda}} \cdot p_j$. We already have $\bar{\lambda}$, which can only be equal to γ ($\bar{\lambda}$ can also be obtained from the definition $\bar{\lambda} = \sum_{j=0}^{+\infty} \lambda_j \cdot p_j$, albeit with some more computations). Thus, we have:

$$r_{2k} = \frac{\lambda}{\gamma} \cdot p_{2k} = \frac{\lambda+\mu}{2\lambda+\mu} \cdot 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (2\lambda+\mu)}, \quad k \geq 0$$

$$r_{2k+1} = \frac{2\lambda}{\gamma} \cdot p_{2k+1} = 2 \cdot \frac{\lambda+\mu}{2\lambda+\mu} \cdot 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda+\mu)}$$

$$= 2^{k+1} \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (2\lambda+\mu)}, \quad k \geq 0$$