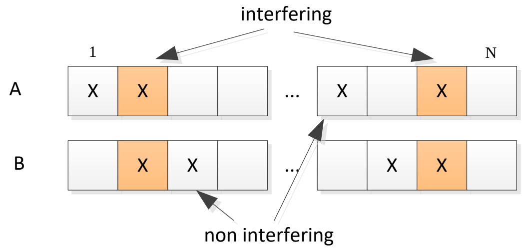
Exercise 1

Consider a system where two entities A and B allocate elements in their own vectors of N elements, numbered from 1 to N. We say that there is *interference* on element j if it is allocated in both A and B's vectors. Assume that each entity allocates its elements *at random*, and *independently*, and call n_A, n_B the *number of elements* allocated by each entity, $0 \le n_x \le N$.



1) Compute the probability that the allocation of A includes element 1.
2) Compute the probability that the allocation of A includes element j.
3) Compute the probability that there is interference on element j. Verify your answer in limit cases.
4) Compute L, U, i.e., the minimum and maximum number of interfering elements among all possible allocations.
5) Compute the probability that the allocations of A and B have exactly k interfering elements, for a generic k, $L \le k \le U$.
6) Assume that $n_A = n_B = n$. Compute the probability that the two allocations are completely overlapping. Find a combinatorial rationale for the result.

Exercise 2

A car repair service company has n repair bays, and expects customers' cars to come in for repair with exponentially distributed interarrivals, at a rate λ. The repair of a car takes an exponentially distributed time with a mean of $\frac{1}{\mu}$. The company wants to man the smallest possible number of repair bays (so as to save money), but knows that its customers find it unacceptable to have to wait.

1) Model the above system as a birth-death process and draw its CTMC.
2) Compute the steady-state probabilities. Express the stability condition.
3) Compute the probability P_{wait} that a car that breaks has to *wait* before entering a repair bay.
4) Assume $\lambda = \mu$. Compute P_{wait} as a function of n and study its behavior with n.
5) Under the above hypothesis, state whether 6 manned repair bays are enough to have P_{wait} smaller than $5 \cdot 10^{-4}$.

It may be useful to observe that $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} = \lim_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]_{x=1} = [e^x]_{x=1} = e$, and that $\sum_{j=0}^{n} \frac{1}{j!} \approx e$ when $n \ge 5$.

Exercise 1 - Solution

1) This is a uniform probability model. Therefore, the answer is $P = \frac{\binom{N-1}{n_A-1}}{\binom{N}{n_A}} = \frac{n_A}{N}$.

2) The answer is the same as before, there being nothing special about any particular element in the vectors.

3) Since the two allocations are independent, the answer is $\frac{n_A}{N} \cdot \frac{n_B}{N} = \frac{n_A \cdot n_B}{N^2}$. The result is *null* if either of the two allocations is null, and it is equal to one only if both entities allocate the whole vector.

4) It is fairly obvious that the number of interfering elements is upper bounded by $U = min(n_A, n_B)$. The lower bound is zero, if $n_A + n_B \leq N$, and $n_A + n_B - N$ otherwise. Thus, $L = max(0, n_A + n_B - N)$.

5) We are in a UPM. The sample space is the set of all possible allocations, whose cardinality is: $|\boldsymbol{S}| = \binom{N}{n_A} \cdot \binom{N}{n_B}$. There are $\binom{N}{k}$ subset of k interfering elements. These must be common to both allocations. This leaves A with $\binom{N-k}{n_A-k}$ ways to allocate the remaining $n_A - k$ non-interfering elements, and B with $\binom{N-n_A}{n_B-k}$ possible ways to allocate the remaining $n_B - k$ (note that the two expressions are different, since we cannot allow interference between the remaining $n_A - k$ of vector A and the remaining $n_B - k$ elements of vector B). By applying the basic principle of counting, we obtain:

$$P = \frac{\binom{N}{k} \cdot \binom{N-k}{n_A-k} \cdot \binom{N-n_A}{n_B-k}}{\binom{N}{n_A} \cdot \binom{N}{n_B}}$$

The alert reader can easily check that – despite the appearances – the above formula is symmetric (i.e., swapping n_A, n_B yields the same result).

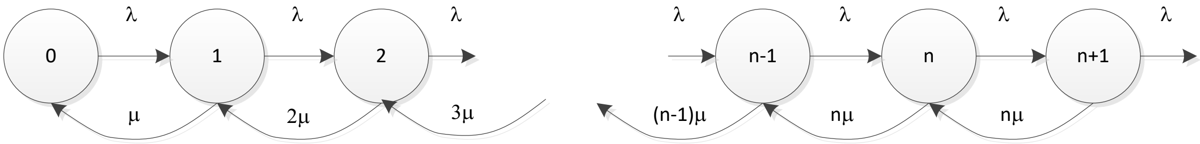
6) From the above, we obtain:

$$P = \frac{\binom{N}{n} \cdot \binom{N-n}{n-n} \cdot \binom{N-n}{n-n}}{\binom{N}{n} \cdot \binom{N}{n}} = \frac{1}{\binom{N}{n}}$$

The result can be easily explained as follows: assume A allocates n elements. Of all the possible allocations at B, which are $\binom{N}{n}$, there is only *one* that has exactly the same elements as the other.

Exercise 2 - Solution

1) The system is an *M*/*M*/*n* one, hence the CTMC is the following:



2) We know from the theory that the system is stable if $\rho = \frac{\lambda}{(n\cdot\mu)} < 1$. This should also emerge from the computation of the steady-state probabilities. The global equilibrium equations are the following:

$$\begin{aligned}
&P_0 \cdot \lambda = P_1 \cdot \mu \\
&P_1 \cdot \lambda = P_2 \cdot 2\mu \\
&\dots \\
&P_{n-1} \cdot \lambda = P_n \cdot n \cdot \mu \\
&P_n \cdot \lambda = P_{n+1} \cdot n \cdot \mu \\
&\dots \\
&P_{n+j} \cdot \lambda = P_{n+j+1} \cdot n \cdot \mu \quad j \geq 0
\end{aligned}$$

From which we get:

$$P_j = \begin{cases} \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \cdot P_0 & j < n \\ \rho^j \cdot \frac{n^n}{n!} \cdot P_0 & j \geq n \end{cases}$$

Hence, the normalization condition is:

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \sum_{j=n}^{\infty} \rho^n \right\} = 1$$

The infinite sum converges if and only if $\rho < 1$, as expected. This said,

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \left[\sum_{j=0}^{\infty} \rho^n - \sum_{j=0}^{n-1} \rho^n \right] \right\} = 1$$

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n\cdot\rho)^n}{1-\rho} \right\} = 1$$

$P_0 = \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n\cdot\rho)^n}{1-\rho} \right\}^{-1}$.

If *n* is large, the following approximation is reasonable: $P_0 = \frac{1}{e^{\frac{\lambda}{\mu}} + \frac{1}{n!} \cdot \frac{(n\cdot\rho)^n}{1-\rho}}$

3) Since the system enjoys the PASTA property, the probability that a car that breaks has to wait before entering service is the probability that $j \geq n$ customers are in the system, i.e. $\sum_{j=n}^{\infty} r_j = \sum_{j=n}^{\infty} P_j$. This can be written as:

$$P_{wait} = \sum_{j=n}^{\infty} P_j = \frac{n^n}{n!} \cdot P_0 \cdot \sum_{j=n}^{\infty} \rho^j = \frac{(n\cdot\rho)^n}{n!\,(1-\rho)} \cdot P_0 = \frac{\frac{(n\cdot\rho)^n}{n!\,(1-\rho)}}{\sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{(n\cdot\rho)^n}{n!\,(1-\rho)}}$$

Again, if n is large, the following approximation is reasonable:

$$P_{wait} \approx \frac{1}{\frac{n!\,(1-\rho)\cdot e^{\frac{\lambda}{\mu}}}{(n\cdot\rho)^n}+1}$$

4) Note that $\lambda = \mu$ implies $n > 1$, otherwise the system is unstable. When $\lambda = \mu$, we get:

$$P_{wait} = \frac{\frac{\left(n\cdot\frac{1}{n}\right)^n}{n!\left(1-\frac{1}{n}\right)}}{\sum_{j=0}^{n-1}\left[\frac{1}{j!}\right]+\frac{\left(n\cdot\frac{1}{n}\right)^n}{n!\left(1-\frac{1}{n}\right)}} = \frac{1}{(n-1)!\,(n-1)\cdot\sum_{j=0}^{n-1}\left[\frac{1}{j!}\right]+1}$$

The above statement is confirmed by the fact that $(n-1)$ appears in the denominator. P_{wait} is obviously decreasing with n (since the denominator increases with n). Moreover, $P_{wait}(n) \geq \frac{1}{(n-1)!(n-1)\cdot e+1}$, with $P_{wait}(n) \approx \frac{1}{(n-1)!(n-1)\cdot e+1}$ when $n \geq 5$. The first few values are reported in the table:

n	Pwait expr.	Numerical value
2	$\frac{1}{3}$	0.333333
3	$\frac{1}{11}$	0.090909
4	$\frac{1}{49}$	0.020408
5	$\frac{1}{261}$	0.003831

5) When $n = 6$, it is $P_{wait}(n) \approx \frac{1}{600\cdot e+1}$. Since $e < 3$, it is $P_{wait}(n) > \frac{1}{1800} > \frac{1}{2000} = 5\cdot 10^{-4}$, so the answer is no.