PECSN 15/06/2022

Exercise 1

Given an exponential RV Y, whose rate is α, consider RV $X = x_m \cdot e^Y$, $x_m > 0$.

1) Compute the interval of values for X, its CDF and its PDF
2) Compute the mean value of X and its variance
3) Is X heavy-tailed? Justify your answer
4) Compute conditional probability $P\{X > a + b | X > a\}$, with a,b positive values. Is the distribution memoryless? What happens to the above probability if $a \to +\infty$?

Exercise 2

Consider a computer system to which jobs are sent through an encrypted channel, with exponential interarrival times at a rate λ. When a *new* job arrives, the system first *decrypts* it. Decryption takes an exponential time, with a mean $\frac{1}{\alpha}$. Decrypted jobs are served in FIFO order with an exponential rate equal to μ. However, the system *cannot* perform any other task when decrypting a new job: either it is accepting/serving jobs, or it is decrypting one. More specifically, any arrival that occurs when the system is decrypting a job is lost, and any ongoing service is suspended and then resumed afterwards.

1) Model the system as a queueing system and draw the transition-rate diagram
2) Find the stability condition and compute the steady-state probabilities. State explicitly their dependence on α, and justify your answer.
3) Compute the mean arrival rate at the system and the tagged-arrival probabilities.
4) Compute the mean number of jobs in the system and the mean response time.

Exercise 1 – Solution

1) The interval of values for X is $x_m, +\infty)$.

$$F_X(x) = P\{X \le x\} = P\{x_m \cdot e^Y \le x\} = P\left\{Y \le log\left(\frac{x}{x_m}\right)\right\} = 1 - e^{-\alpha \cdot log\left(\frac{x}{x_m}\right)} = 1 - \left(\frac{x_m}{x}\right)^{\alpha}$$

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \frac{\alpha \cdot {x_m}^{\alpha}}{x^{\alpha+1}}$$

2) The first two moments are:

$$E[X] = \int_{x_m}^{+\infty} x \cdot \frac{\alpha \cdot {x_m}^{\alpha}}{x^{\alpha+1}} dx = \frac{\alpha \cdot {x_m}^{\alpha}}{1-\alpha} \cdot \left[x^{(1-\alpha)}\right]_{x_m}^{+\infty} = \begin{cases} \frac{\alpha}{\alpha-1} \cdot x_m & \alpha > 1 \\ +\infty & otherwise \end{cases}$$

$$E[X^2] = \int_{x_m}^{+\infty} x^2 \cdot \frac{\alpha \cdot {x_m}^{\alpha}}{x^{\alpha+1}} dx == \alpha \cdot {x_m}^{\alpha} \int_{x_m}^{+\infty} x^{-\alpha+1} dx = \frac{\alpha \cdot {x_m}^{\alpha}}{2-\alpha} \cdot \left[x^{(2-\alpha)}\right]_{x_m}^{+\infty}$$

$$= \begin{cases} \frac{\alpha}{\alpha-2} \cdot {x_m}^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

From the above, we obtain:

$$\sigma^2 = E[X^2] - E[X]^2 = \begin{cases} \frac{\alpha}{\alpha-2} \cdot {x_m}^2 - \frac{\alpha^2}{(\alpha-1)^2} \cdot {x_m}^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

$$= \begin{cases} \frac{\alpha}{(\alpha-2)} \cdot \left(\frac{x_m}{\alpha-1}\right)^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

3) The definition of heavy-tailed distribution is: $\forall \lambda > 0, \quad \lim_{x\to\infty} e^{\lambda x} \cdot \left(1 - F(x)\right) = \infty$

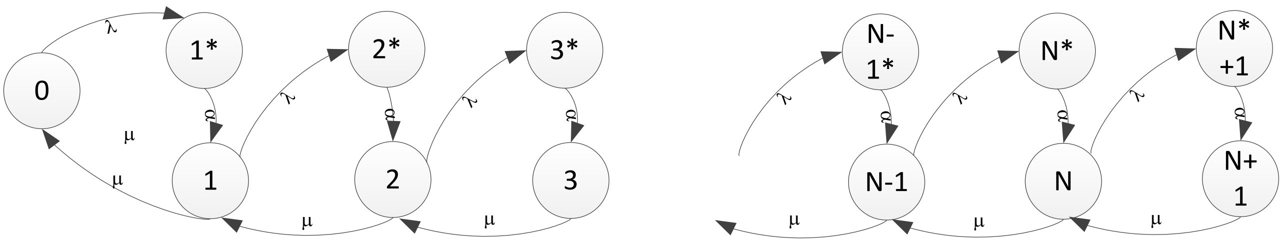
In this case, we have $\forall \lambda > 0, \quad \lim_{x\to\infty} e^{\lambda x} \cdot \left(\frac{x_m}{x}\right)^{\alpha} = +\infty$, hence the above distribution is heavy-tailed.

4) The above conditional probability is computed as follows:

$P\{X > a + b | X > a\} = \frac{P\{X>a+b\}}{P\{X>a\}} = \left(\frac{x_m}{a+b}\right)^{\alpha} \cdot \left(\frac{a}{x_m}\right)^{\alpha} = \left(\frac{a}{a+b}\right)^{\alpha}$. The distribution is not memoryless, since the above is not equal to $P\{X > b\} = \left(\frac{x_m}{b}\right)^{\alpha}$. When $a \to +\infty$, the conditional probability approaches 1. This means that if you have exceeded a very large value, then the probability that you will exceed an even larger one approaches 1.

Exercise 2 - Solution

1) The CTMC is the following. Note that the state of the system cannot be described using only the number of jobs, since the fact that the system is *decrypting* or *serving* jobs is also relevant. Starred states are those when the system is decrypting. Furthermore, you can observe that when $\alpha \to \infty$ (i.e., decryption is much faster than the rate of arrival or service) the two states in the same column collapse to one, and the system becomes an M/M/1.



2) Using local equilibrium equations, one can easily check that $p_j \cdot \lambda = p_{j+1} \cdot \mu$, hence $p_j = \left(\frac{\lambda}{\mu}\right)^j \cdot p_0$, $j \geq 0$. Moreover, using global equilibrium equation around the decrypting states, one obtains $p_j{}^* \cdot \alpha = p_{j-1} \cdot \lambda, j \geq 1$, hence $p_j{}^* = \frac{\mu}{\alpha} \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot p_0,\ j \geq 1$.
The normalization condition then reads $p_0 \left[\sum_{j=0}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j + \frac{\mu}{\alpha} \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j\right] = 1$. Call $\rho = \frac{\lambda}{\mu}$, the stability condition is then $\rho < 1$, and it is independent of α. This is as expected, since during starred states the system is neither accepting nor serving jobs, so the decryption rate cannot have any influence on stability. After a few straightforward computations, one obtains:
$p_j = (1 - \rho) \cdot \rho^j \cdot \frac{\alpha}{\lambda+\alpha},\ j \geq 0$.
$p_j{}^* = (1 - \rho) \cdot \rho^j \cdot \frac{\mu}{\lambda+\alpha}, j \geq 1$.
3) The average arrival rate in the system is $\overline{\lambda} = \lambda \cdot \sum_{j=0}^{+\infty} p_j + 0 \cdot \sum_{j=0}^{+\infty} p_j{}^* = \lambda \cdot \frac{\alpha}{\lambda+\alpha}$. The tagged arrival probabilities are $r_j = p_j \cdot \frac{\lambda}{\overline{\lambda}} = (1 - \rho) \cdot \rho^j, r_j{}^* = 0$.
4) The mean number of jobs in the system (which includes a job being decrypted, if there is one) is:

$$E[N] = \sum_{j=1}^{+\infty} j \cdot \left(p_j + p_j{}^*\right) = \frac{\mu + \alpha}{\lambda + \alpha} \cdot \sum_{j=1}^{+\infty} j \cdot (1 - \rho) \cdot \rho^j = \frac{\mu + \alpha}{\lambda + \alpha} \cdot \frac{\rho}{1 - \rho}$$

The mean traversal time for a job can be found by applying Little's Law, and it is:

$$E[R] = \frac{E[N]}{\overline{\lambda}} = \frac{\mu + \alpha}{\lambda + \alpha} \cdot \frac{\rho}{1 - \rho} \cdot \frac{\lambda + \alpha}{\lambda \cdot \alpha} = \frac{\mu + \alpha}{\alpha} \cdot \frac{1}{\mu - \lambda}$$