PECSN, 23/2/2022

Exercise 1

Consider function $f(x,y) = \begin{cases} e^{3x-\frac{y}{k}} & x \le 0, y \ge 0 \\ 0 & otherwise \end{cases}$, with $k > 0$.

1) Determine k so that the above is a JPDF for RVs X and Y
2) Compute the single PDFs of RVs X and Y
3) Are X and Y independent RVs? Are they identically distributed?
4) Compute the 95th percentile of X and the median of Y.

Exercise 2

Consider a manufacturing plant where K products are being processed at any time. As soon as a product is completed, another one is started. New products may require *one to four* stages of processing, with the same probability. When a new product arrives, it is examined at SC 5, and then sent to SCs 1,2,3,4, depending on the number of stages of processing it requires. Each SC, on terminating its processing of a product, sends it to the *next* SC for processing (i.e., SC 1 sends it to SC 2, etc.). A product that reaches SC 5 again is considered completed, and it leaves the plant (to be replaced by a new product, so as to keep K constant). Assume that the service rates at SCs 1-5 are all the same and equal to μ, and all SCs can be modeled as M/M/1 systems.

1) Model the system as a queueing network
2) Solve the routing equation system and compute the SS probabilities in its general form

Assume $K = 4$

3) Compute the normalizing constant using Buzen's algorithm
4) Compute the utilization of SC j, $j = 1, \dots, 5$
5) Compute the probability that all jobs are on a single SC

Exercise 1 – Solution

1) In order for f to be a JPDF, the normalization condition must hold, hence:

$$\int_0^{+\infty}\int_{-\infty}^{0} e^{3x-\frac{y}{k}}\,dxdy = 1$$
$$\int_0^{+\infty} e^{-\frac{y}{k}}\left[\int_{-\infty}^{0} e^{3x}dx\right]dy = 1$$
$$\frac{1}{3}\int_0^{+\infty} e^{-\frac{y}{k}}\,dy = 1$$
$$\frac{k}{3} = 1$$
$$k = 3$$

2) The PDFs for the single variables are:

$$f_X(x) = \int_0^{+\infty} e^{3x-\frac{y}{3}}\,dy = e^{3x}\int_0^{+\infty} e^{-\frac{y}{3}}dy = 3\cdot e^{3x}$$

$$f_Y(y) = \int_{-\infty}^{0} e^{3x-\frac{y}{3}}\,dx = e^{-\frac{y}{3}}\int_{-\infty}^{0} e^{3x}\,dx = \frac{1}{3}\cdot e^{-\frac{y}{3}}$$

3) The two PDFs are clearly independent, since $f(x,y) = f_X(x)\cdot f_Y(y)$. However, they are not equal, hence the two RVs are not IID.

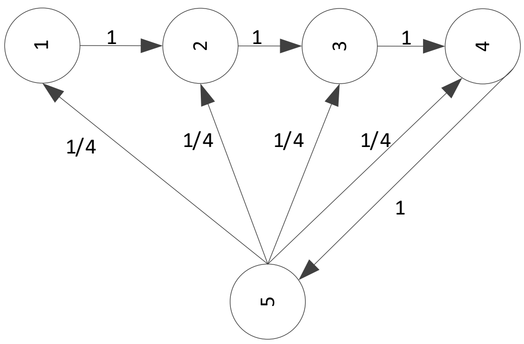
4) The 95th percentile of X can be found by imposing that $F_X(\pi_{95}) = \int_{-\infty}^{\pi_{95}} f_X(x)dx = 0.95$, i.e.

$$\int_{-\infty}^{\pi_{95}} 3e^{3x}dx = 0.95$$
$$e^{3\pi_{95}} = 0.95$$
$$\pi_{95} = \frac{1}{3}log(0.95)$$

Similarly, the median of Y is found by imposing $F_Y(\pi_{50}) = 0.5$, which yields $\pi_{50} = -3\,log(0.5)$.

Exercise 2

The network is a closed Jackson's network. A routing diagram is the following:



The routing matrix is :

$$\boldsymbol{\Pi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Hence the linear system to be solved is the following:

$$\begin{cases} \lambda_1 = \lambda_5/4 \\ \lambda_2 = \lambda_1 + \lambda_5/4 \\ \lambda_3 = \lambda_2 + \lambda_5/4 \\ \lambda_4 = \lambda_3 + \lambda_5/4 \\ \lambda_5 = \lambda_4 \end{cases}$$

A solution of the above is $\boldsymbol{e}^T = \begin{bmatrix} \frac{\mu}{4} & \frac{2\mu}{4} & \frac{3\mu}{4} & \mu & \mu \end{bmatrix}^T$, from which we obtain $\boldsymbol{\rho}^T = \begin{bmatrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 1 \end{bmatrix}^T$.

From this, we obtain the following SS probabilities:

$$p_{\underline{n}} = p(n_1, n_2, n_3, n_4, n_5) = \frac{1}{G(5,K)} \cdot \left[\frac{1}{4^{n_1}} \cdot \frac{2^{n_2}}{4^{n_2}} \cdot \frac{3^{n_3}}{4^{n_3}} \right] = \frac{1}{G(5,K)} \cdot \frac{3^{n_3}}{2^{2n_1+n_2+2n_3}}$$

Constant $G(5,4)$ can be computed through Buzen's algorithm.

rho	1/4	2/4	3/4	1	1
s.c.	1	2	3	4	5
jobs					
0	1	1	1	1	1
1	1/4	3/4	6/4	10/4	14/4
2	1/16	7/16	25/16	65/16	121/16
3	1/64	15/64	90/64	350/64	834/64
4	1/256	31/256	301/256	1701/256	5037/256

$$G(5,4) = \frac{5037}{256} \cong 19.68$$

The utilization of the SCs is $U_j = \rho_j \cdot \frac{G(5,3)}{G(5,4)} = \rho_j \cdot \frac{834}{64} \cdot \frac{256}{5037} = \rho_j \cdot \frac{3336}{5037} \cong 0.66 \cdot \rho_j$. Thus, the utilizations are: $\boldsymbol{U}^T = \begin{bmatrix} \frac{834}{5037} & \frac{1668}{5037} & \frac{2502}{5037} & \frac{3336}{5037} & \frac{3336}{5037} \end{bmatrix}^T \cong [0.17 \quad 0.33 \quad 0.5 \quad 0.66 \quad 0.66]^T$.

The probability that all jobs are on SC j is:

- $j = 1: p_{\underline{n}} = p(4,0,0,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{8+0+0}} = \frac{256}{5037} \cdot \frac{1}{256} = \frac{1}{5037}$
- $j = 2: p_{\underline{n}} = p(0,4,0,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+4+0}} = \frac{256}{5037} \cdot \frac{1}{16} = \frac{16}{5037}$
- $j = 3: p_{\underline{n}} = p(0,0,4,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^4}{2^{0+0+8}} = \frac{256}{5037} \cdot \frac{81}{256} = \frac{81}{5037}$
- $j = 4: p_{\underline{n}} = p(0,0,0,4,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+0+0}} = \frac{256}{5037}$
- $j = 5: p_{\underline{n}} = p(0,0,0,0,4) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+0+0}} = \frac{256}{5037}$

Therefore, the probabilities that all jobs are at a single SC is the sum of the above, i.e.

$$p = \frac{1 + 16 + 81 + 256 + 256}{5037} = \frac{610}{5037} \cong 0.12$$