Exercise 1

Consider a set of n six-faced fair dice, which are thrown independently. Call X_n the random variable denoting the *maximum* value obtained in the throw of all the dice.

1) Compute (as a formula) the PMF $f_2(x)$ of X_2;
2) Compute (as a formula) the PMF $f_3(x)$ of X_3. Hint: compute the CDF *first*;
3) Compute $f_n(x)$ for any number of dice n. Verify the normalization condition;
4) Draw a qualitative diagram of $f_n(x)$. Explain your findings;
5) Find the smallest number of dice *n* such that you exceed 90% probability that you will get a 6 as the maximum.

Exercise 2

A network buffer has enough space for three packets. It employs a *gated* policy, meaning that it only accepts ingresses when the system is *empty*. Ingresses come in the form of *messages*, each one containing *one, two or three* packets, with probability q_1, q_2, q_3 respectively. The interarrival time of messages is an exponentially distributed variable with a mean equal to $\frac{1}{\lambda}$. The buffer processes *packets* (not messages), and the service time of a packet is an exponentially distributed variable with a mean equal to $\frac{1}{\mu}$.

1) Model the system and draw the CTMC (or transition rate diagram);
2) Compute the steady-state probabilities and the stability condition;
3) Determine which number of packets in the system is the most likely at the steady-state;
4) Compute the mean number of packets in the system and in the queue. State the conditions under which the mean number of packets in the system is larger than one;
5) Compute the system utilization.

Exercise 1 – Solution

1) Since the dice are fair, each outcome $\{d_1, d_2\}$ is equally likely. Therefore, you can apply the basic principle of counting and obtain:
   - $f_2(1) = \frac{1}{36}$, (the "good" outcome is {1,1})
   - $f_2(2) = \frac{3}{36}$, (the "good" outcomes are {1,2}, {2,1}, {2,2})
   - $f_2(3) = \frac{5}{36}$, etc.,

   Which leads to $f_2(x) = \frac{2x-1}{36}$, $1 \le x \le 6$.
2) The same reasoning applies. However, in order to compute (say) $f_3(4)$ we would need to count really many outcomes (i.e., all the permutations of {1,1,4}, {1,2,4}, etc.), and this is cumbersome, hence error-prone. Following the hint, it is considerably easier to compute $F_3(x) = P\{max[d_1, d_2, d_3] \le x\}$, $1 \le x \le 6$. In fact:
   - $F_3(1) = \frac{1}{6^3}\left(= \frac{1^3}{6^3}\right)$, (the only "good" outcome being {1,1,1})
   - $F_3(2) = \frac{2^3}{6^3}$, since the "good" outcomes are all the sequences $\{d_1, d_2, d_3\}$ where $d_i \le 2$.
   - Similarly, $F_3(k) = \frac{k^3}{6^3}$, $1 \le x \le 6$.

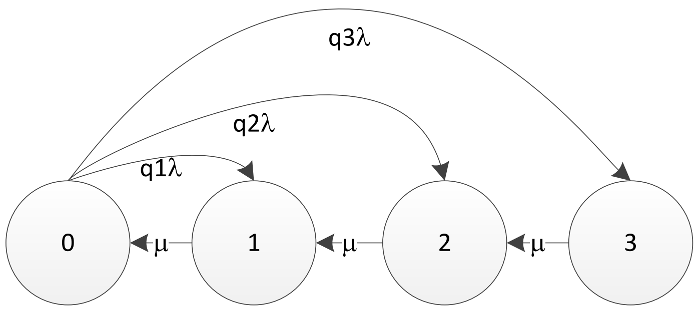
   Therefore, $f_3(k) = F_3(k) - F_3(k-1) = \frac{k^3-(k-1)^3}{6^3}$.
3) The above expressions immediately generalize to $F_n(k) = \frac{k^n}{6^n}$, and $f_n(k) = F_n(k) - F_n(k-1) = \frac{k^n-(k-1)^n}{6^n}$, $1 \le x \le 6$. The normalization condition is verified, since $F_n(6) = \frac{6^n}{6^n} = 1 \quad \forall n$. Note that the above formulas are compatible with the one found at point 1), once the required (obvious) simplifications are performed.
4) $f_n(k)$ is increasing with *k*, for any *n*. It is easy to see that:

$$\lim_{n\to\infty} f_n(k) = \lim_{n\to\infty} \frac{k^n - (k-1)^n}{6^n} = \lim_{n\to\infty} \left[\left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n\right] = \begin{cases} 0 & k < 6 \\ 1 & k = 6 \end{cases}$$

   Therefore, as *n* grows large, the diagram tends to be *flat* towards zero for $k < 6$ and have a spike in *k*=6. This is because, as the number of dice grows large, it becomes increasingly unlikely that the maximum value obtained when throwing *n* dice is smaller than 6.
5) It is $f_n(6) = 0.9$, i.e. $1 - \left(\frac{5}{6}\right)^n = 0.9$. This yields $\left(\frac{5}{6}\right)^n = 0.1$, i.e. $n[log_{10} 5 - log_{10} 6] = -1$, i.e. $n = \left\lceil \frac{1}{log_{10} 6 - log_{10} 5} \right\rceil = 13$.

Exercise 2 - Solution

1) The CTMC is as follows. Note that $q_1 + q_2 + q_3 = 1$, obviously.



2) The steady state probabilities are computed by writing down the global equilibrium equations:

$$P_0 \cdot (q_1 + q_2 + q_3) \cdot \lambda = P_1 \cdot \mu$$
$$P_1 \cdot \mu = P_2 \cdot \mu + P_0 \cdot q_1 \cdot \lambda$$
$$P_2 \cdot \mu = P_3 \cdot \mu + P_0 \cdot q_2 \cdot \lambda$$
$$P_3 \cdot \mu = P_0 \cdot q_3 \cdot \mu$$

One of the above equations is redundant. The system is always stable, since it has a finite queue. By solving the above system, we obtain:

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (q_1 + 2q_2 + 3q_3)} = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (1 + q_2 + 2q_3)}$$
$$P_1 = \frac{\lambda}{\mu} \cdot P_0 = \frac{\lambda}{\mu} \cdot (q_1 + q_2 + q_3) P_0$$
$$P_2 = \frac{\lambda}{\mu}(1 - q_1) \cdot P_0 = \frac{\lambda}{\mu}(q_2 + q_3) \cdot P_0$$
$$P_3 = \frac{\lambda}{\mu} q_3 \cdot P_0$$

3) From the above, it is clear that $P_1 \geq P_2 \geq P_3$, whatever the values q_i. Whether $P_0 > P_1$ or vice versa instead depends on whether $\lambda > \mu$. Thus, the answer is P_0 if $\lambda > \mu$, and P_1 otherwise.

4) It is:

$$E[N] = \sum_{n=1}^{3} n \cdot P_n = \frac{\lambda}{\mu} \cdot P_0 \cdot [1 \cdot (q_1 + q_2 + q_3) + 2(q_2 + q_3) + 3q_3]$$
$$= \frac{q_1 + 3q_2 + 6q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{1 + 2q_2 + 5q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$
$$E[N_q] = 1 \cdot P_2 + 2 \cdot P_3 = \frac{\lambda}{\mu} \cdot P_0 \cdot (q_2 + 3q_3) = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$

In order to have $E[N] > 1$ we need to have $q_1 + 3q_2 + 6q_3 > \frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3$, which translates to
$q_2 + 3q_3 > \frac{\mu}{\lambda}$.

5) The system utilization is $1 - P_0$, i.e.

$$U = \frac{q_1 + 2q_2 + 3q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{1 + q_2 + 2q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$