Exercise 1

A safety-critical distributed application exchanges messages via a noisy channel. The sender keeps sending copies of the same message until it receives k ACKs from the receiver. The protocol is run on a slotted network: on every slot the sender sends a new copy of the message, and it receives an ACK with probability p. Transmissions in different slots are independent.
Call X_k the random variable that counts the number of slots until the communication is considered completed by the sender.

1) Find the set of values for X_k;
2) Compute the PMF of X_1, and explain your result;
3) Compute the PMF of X_2;
4) Compute the PMF of X_3 and generalize the above result to any value of k;
5) Compute the mean value of X_k.

Exercise 2

Consider a processing system where jobs arrive with exponential interarrival times with a mean $1/\lambda$ and request a service time which is also exponentially distributed with a mean $1/\mu$. To save power, the system server is switched *off* whenever the system is empty, and it is switched back *on* again when k jobs are in the system.

1) Model the system and draw the CTMC. Verify your model in the limit case $k = 1$;
2) Write the equilibrium equations, find the stability condition, and state explicitly its dependence on k. Justify your answer;
3) Compute the probability of having n jobs in the system at the steady state;
4) Compute the server utilization and the throughput.

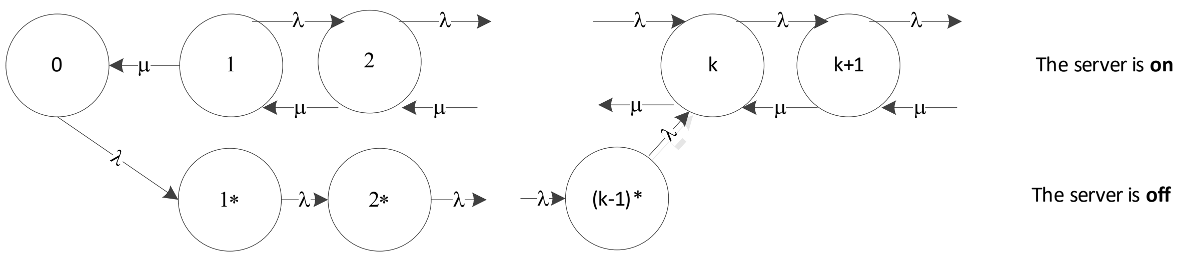
Exercise 1 – Solution

This is a repeated-trial experiment, with slots being trials.

1) The set of values for X_k is $k, +\infty)$.
2) X_1 is the number of trials to the first success, hence is a geometric RV: $P\{X_1 = j\} = (1-p)^{j-1} \cdot p$. Note that there is only one sequence of Bernoullian outcomes that terminates the repeated-trial experiment on trial j.
3) X_2 is the number of trials to the *second* success. $P\{X_2 = j\}$ can be computed by reasoning in terms of a composite experiment: one experiment consists in obtaining a success at the j^{th} trial, and the other consists in obtaining *one* success *in* $j-1$ trials. The first probability is equal to p, whereas the second one is equal to $\binom{j-1}{1}(1-p)^{j-2} \cdot p = (j-1) \cdot (1-p)^{j-2} \cdot p$. Therefore, we have: $P\{X_2 = j\} = (j-1) \cdot (1-p)^{j-2} \cdot p^2$.
4) X_3 is the number of trials to the *third* success. Again, this can be split into the probability of having a success on the j^{th} trial, times the probability of having $3-1=2$ successes in the first $j-1$ trials. The result is therefore: $P\{X_3 = j\} = p \cdot \left[\binom{j-1}{2} \cdot (1-p)^{j-3} \cdot p^2\right] = \binom{j-1}{2} \cdot (1-p)^{j-3} \cdot p^3$. From the above, it is clear that the correct expression is $P\{X_k = j\} = p \cdot \left[\binom{j-1}{k-1} \cdot (1-p)^{j-k} \cdot p^{k-1}\right] = \binom{j-1}{k-1} \cdot (1-p)^{j-k} \cdot p^k, j \geq k$.
5) One can observe that X_k is the sum of k geometric RVs, each one having a mean $1/p$. Therefore, it is $E[X_k] = k/p$. The same conclusion can be reached even without the above observation, via a modicum of algebra.

Exercise 2 - Solution

1) When j jobs are in the system, $0 < j < k$, the server may be either *on* or *off*. In either case, the system behaves differently. Thus, we need *two states* for each number of jobs in the system $0 < j < k$. The CTMC is therefore the following:



Starred states are traversed when the server is switched off. When $k = 1$, the interval $0 < j < k$ is empty, hence the system behaves like an M/M/1 and there are no starred states.

2) The stability condition should not depend on k. In fact, if the server is faster than the arrivals, the fact that k jobs are allowed to queue up does not thwart its ability to empty the queue. This must be confirmed by the computations.

The global equilibrium equations for the starred states are: $\lambda \cdot P_0 = \lambda \cdot P_{1^*} = \ldots = \lambda \cdot P_{(k-1)^*}$. Therefore, it is $P_{j^*} = P_0$, $1 < j \leq k-1$.

Moreover, it is easy to write *local* equilibrium equations along "vertical" cuts, as follows:

$$\lambda \cdot P_o = \mu \cdot P_1$$
$$\lambda \cdot \left(P_j + P_{j^*}\right) = \mu \cdot P_{j+1} \quad 1 \leq j \leq k-1$$
$$\lambda \cdot P_j = \mu \cdot P_{j+1} \qquad j \geq k$$

Setting $\rho = \frac{\lambda}{\mu}$ for ease of writing, we get:

$$\begin{aligned} &P_1 = \rho \cdot P_o \\ &P_{j+1} = \rho \cdot (P_j + P_0) \quad 1 \le j \le k-1 \\ &P_{j+1} = \rho \cdot P_j \qquad\qquad j \ge k \end{aligned}$$

Therefore, we get:

$$\begin{aligned} &P_1 = \rho \cdot P_o \\ &P_2 = \rho \cdot (P_1 + P_0) = (\rho^2 + \rho) \cdot P_0 \\ &P_3 = \rho \cdot (P_2 + P_0) = (\rho^3 + \rho^2 + \rho) \cdot P_0 \\ &\dots \\ &P_j = \left(\sum_{i=1}^{j} \rho^i\right) \cdot P_0 = \frac{\rho - \rho^{j+1}}{1-\rho} \cdot P_0 \qquad 1 \le j \le k \\ &P_j = \rho^{j-k} \cdot P_k = \rho^{j-k} \cdot \frac{\rho - \rho^{k+1}}{1-\rho} \cdot P_0 \qquad j > k \end{aligned}$$

[The last two expressions are correct only if $\rho \neq 1$. This condition is necessary for stability in any case.]

The normalization condition is therefore the following:

$$P_0 + \sum_{j=1}^{k-1} P_{j^*} + \sum_{j=1}^{k} P_j + \sum_{j=k+1}^{\infty} P_j = 1$$

Which can be rewritten as:

$$P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho - \rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \sum_{j=1}^{\infty} \rho^j\right] = 1$$

From the latter it clearly appears that the stability condition is $\rho < 1$. In fact, the first two terms are finite. The condition is thus independent of k, as was expected. Through a few algebraic manipulations, we get:

$$\begin{aligned} &P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho - \rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \sum_{j=1}^{\infty} \rho^j\right] = \\ &P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho}{1-\rho} - \sum_{j=1}^{k} \frac{\rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \frac{\rho}{1-\rho}\right] = \\ &P_0 \cdot \left[k + k \cdot \frac{\rho}{1-\rho} - \frac{\rho}{1-\rho} \cdot \frac{\rho - \rho^{k+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \frac{\rho}{1-\rho}\right] = \\ &P_0 \cdot \frac{k}{1-\rho} = 1 \end{aligned}$$

3) From the above expressions we obtain:

$$\begin{aligned} &P_0 = \frac{1-\rho}{k} \\ &P_{j^*} = \frac{1-\rho}{k} \qquad 1 \le j \le k-1 \\ &P_j = \rho \cdot \frac{1-\rho^j}{k} \qquad 1 \le j \le k \end{aligned}$$

$$P_j = \rho^{j-k+1} \cdot \frac{1-\rho^k}{k} \qquad j > k$$

(Note that, when $k = 1$, we get the usual M/M/1 steady-state probabilities). Call π_j the SS probability to have j jobs in the system. It is:

$$\pi_j = \begin{cases} P_j & j = 0 \\ P_j + P_{j^*} & 1 \leq j \leq k-1 \\ P_j & j \geq k \end{cases}$$

Hence:

$$\pi_j = \begin{cases} \frac{1-\rho}{k} & j = 0 \\ \frac{1-\rho^{j+1}}{k} & 1 \leq j \leq k-1 \\ \rho^{j-k+1} \cdot \frac{1-\rho^k}{k} & j \geq k \end{cases}$$

4) When the system is stable, it is $\gamma = \lambda$. Since $\gamma = U \cdot \mu$, we get $U = \rho$. The same result can be obtained by observing that:

$$U = \sum_{j=1}^{+\infty} P_j = 1 - P_0 - \sum_{j=1}^{k-1} P_j^* = 1 - k \cdot \frac{1-\rho}{k} = \rho$$