Exercise 1

Consider two *independent* continuous RVs *X* and *Y*, whose CDFs are the following:

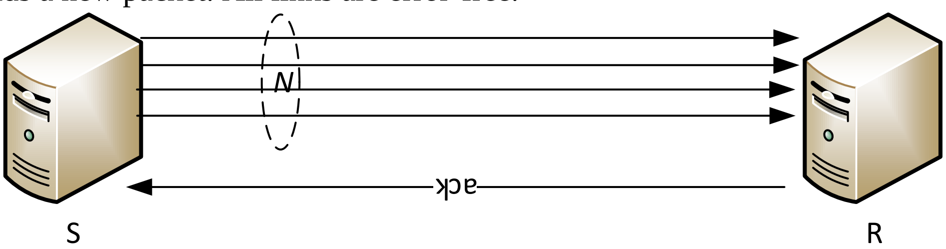
$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/3 & 0 \le x \le 3 \\ 1 & x > 3 \end{cases}, \quad F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \le y \le 2 \\ 1 & y > 2 \end{cases}$$

1) Compute and draw the PDF of both RVs.
2) Draw the region *R* of the Cartesian plane x, y where the JPDF $f(x,y)$ is non null.
3) Identify the range of values for RV Y/X .
4) Let *w* be a value in the interval computed at bullet 3). Plot the region $C_w \subseteq R$, where inequality $Y/X \le w$ holds, on the Cartesian plane x, y. [Hint: try at least $w = 1/2$, $w = 1$].
5) Compute $P\{(X,Y) \in C_w\} = P\{Y/X \le w\} = F_{Y/X}(w)$.
6) Compute and draw the PDF of Y/X .

Exercise 2

Consider a network where a sender S sends packets to a receiver R, using *N* parallel forward links. On sending a packet, S chooses one forward link *at random* and sends the packet to it. Each forward link has a FIFO queue to buffer packets. On receipt of a packet (from any forward link), R sends an *ack* message through the return link. The transmission of both a packet and an ack takes an exponential time with a mean $1/\mu$.

The above communication is subject to a *flow-control protocol*. Flow is controlled by allowing at most *K* packets to be un-acked at any time. Therefore, at the beginning S generates *K* packets, sends each to a forward link at random, and then stops and waits for an ack to come back. Whenever an ack returns, S sends a new packet. All links are error-free.



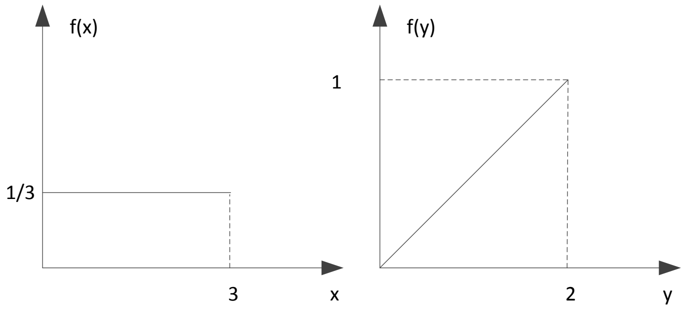
1) Model the above system as a Closed Jackson's Network.
2) Solve the routing equation system and compute the values of ρ_i.
3) Compute the steady-state probabilities as a function of *N* and *K* and interpret the result.

Assume *N*=3 and *K*=6

4) Compute G(M,K) using Buzen's algorithm
5) Compute the probability that S cannot send a packet due to flow control, even though all forward links are idle.
6) Compute the probability that all the forward links are simultaneously busy

Exercise 1 - Solution

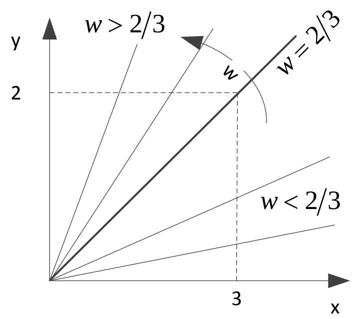
1) X is clearly a uniform RV, hence $f(x)=1/3$, for $0\le x\le 3$, whereas Y's PDF is $f(y)=y/2$, for $0\le y\le 2$. The graphs of the PDFs are the following:



2) The region of the Cartesian plane where the JPDF $f(x,y)=f(x)\cdot f(y)$ is non null is the one where the couple of RV (X,Y) may take on values, i.e. box $R\equiv[0\le x\le 3]\times[0\le y\le 2]$.

3) Since X and Y can only assume nonnegative values, Y/X can only assume nonnegative values. Moreover, since X's infimum is zero, then Y/X takes on values in $[0,+\infty[$.

4) Set C_w is $C_w=\{(x,y):(x,y)\in R,\ y/x\le w\}$, and it includes the area *below* each of the lines drawn in the figure, whose slope is w. Set C_w covers a triangle if $0\le w\le 2/3$, and a trapezoid if $w>2/3$.



5) It is $P\{(X,Y)\in C_w\}=\iint_{C_w} f(x,y)dxdy$, and $f(x,y)=f(x)\cdot f(y)=y/6$ for $(x,y)\in R$. We thus need to integrate $f(x,y)$ on set C_w. We need to distinguish the cases $0\le w\le 2/3$ and $w>2/3$.

a) $0\le w\le 2/3$. We integrate y from 0 to $w\cdot x$, and x from 0 to 3.

$$P\{(X,Y)\in C_w\}=\iint_{C_w}\frac{y}{6}\,dx\,dy=\int_0^3\left[\int_0^{w\cdot x}\frac{y}{6}dy\right]dx=\int_0^3\left[\frac{w^2x^2}{12}\right]dx=\frac{3}{4}w^2$$

b) $w > 2/3$. In this case, it is quicker to use the complement, i.e. to exclude the upper triangular region.

We need to invert the order of integration of the variables.

$$P\{(X,Y)\in C_w\} = \iint_{C_w} \frac{y}{6}\,dx\,dy = 1-\int_0^2\int_0^{y/w}\frac{y}{6}dxdy == 1-\int_0^2 \frac{y^2}{6w}dy = 1-\left[\frac{y^3}{18w}\right]_0^2 = 1-\frac{4}{9w}$$

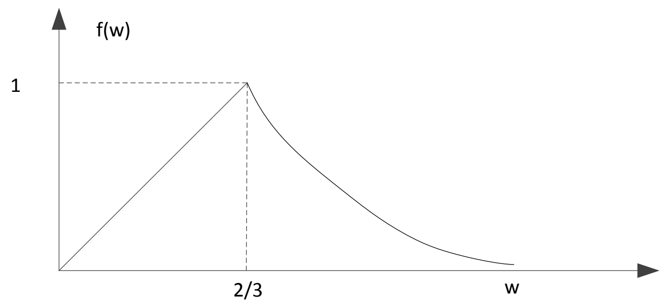
6) Since $P\{(X,Y)\in C_w\} = P\{Y/X \le w\} = F_{Y/X}(w)$, we have:

$$F_{Y/X}(w) = \begin{cases} \frac{3}{4}w^2 & 0 \le w \le 2/3 \\ 1-\frac{4}{9w} & w > 2/3 \end{cases}.$$

Note that $F_{Y/X}(w)$ is continuous, and $\lim_{w\to+\infty} F_{Y/X}(w) = 1$. From that, we get:

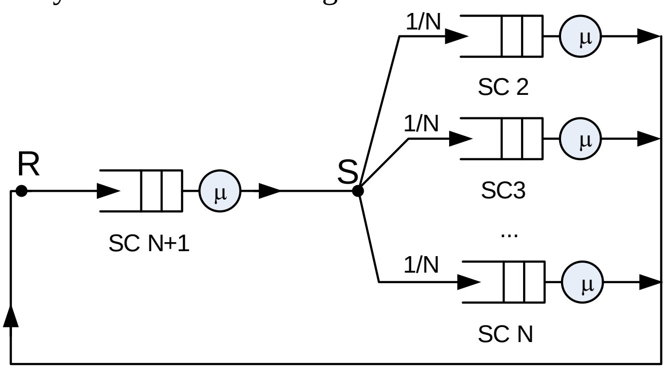
$$f_{Y/X}(w) = \begin{cases} \frac{3}{2}w & 0 \le w \le 2/3 \\ \frac{4}{9w^2} & w > 2/3 \end{cases}$$

The graph of the PDF is the following:



Exercise 2 - Solution

1) Let SCs 1 to N be each output interface at S, and let SC N+1=M be the output interface at R. Then a CJN model of the above system is the following:



2) The routing matrix is the following: $\underline{\mathbf{\Pi}} = \begin{bmatrix} 0 & 0 & ... & 1 \\ 0 & 0 & ... & 1 \\ ... & ... & ... & ... \\ 1/N & 1/N & ... & 0 \end{bmatrix}$

And the equations are $\lambda_i = \lambda_{N+1}/N$, $1 \le i \le N$. It is fairly obvious that solutions are of the form $\underline{\mathbf{e}}^T = [e, e, ..., e, N \cdot e]^T$. Hence, we select $e = \mu$ and obtain $\underline{\boldsymbol{\rho}}^T = [1,1,....,1,N]^T$.

3) The SS probabilities are $P(n_1, n_2, ..., n_{N+1})|_{\sum_{i=1}^{N+1} n_i = K} = \frac{1}{G(N+1,K)} \cdot \prod_{i=1}^{N+1} \rho_i^{n_i} = \frac{1}{G(N+1,K)} \cdot N^{n_{N+1}}$.

Since the above expression only depends on n_{N+1}, this means that any distribution of packets on the forward link queues having the same number of outstanding acks n_{N+1} is equally likely.

4) Buzen's algorithm yields the following table:

rho	1	1	1	3
s.c.	**1**	**2**	**3**	**4**
jobs				
0	1	1	1	1
1	1	2	3	6
2	1	3	6	24
3	1	4	10	82
4	1	5	15	261
5	1	6	21	804
6	1	7	28	2440

Hence $G(4,6) = 2440$.

Note that, due to the findings at point 3), G could also be computed using the normalization condition instead of Buzen's algorithm:

$$\sum_{\bar{n} \in \mathbf{e}} P(n_1, n_2, ..., n_{N+1}) = \sum_{\bar{n} \in \mathbf{e}} \left(\frac{1}{G(N+1,K)} \cdot N^{n_{N+1}} \right) = 1, \text{ from which:}$$

$$G(N+1,K) = \sum_{\bar{n} \in \mathbf{e}} N^{n_{N+1}} = \sum_{n_{N+1}=0}^{K} \binom{K - n_{N+1} + N - 1}{N-1} N^{n_{N+1}}.$$

The binomial coefficient is the number of ways to split $K - n_{N+1}$ jobs (i.e., those that are not on SC N+1) among N SCs, and can be found on the course notes. The two methods obviously yield the same result.

5) The probability that the sender cannot send a packet due to flow control is the probability that K *acks* are queued at the return link, hence $p_{block} = \frac{1}{G(N+1,K)} \cdot N^K = \frac{3^6}{2440} = \frac{729}{2440} \approx 0.3$

6) The probability that all the forward links are simultaneously busy is:

$$p_{busy} = \frac{G(M, K-N)}{G(M,K)} \prod_{i=1}^{N} \rho_i^1 = \frac{G(4,3)}{G(4,6)} = \frac{82}{2440} \approx 0.033.$$