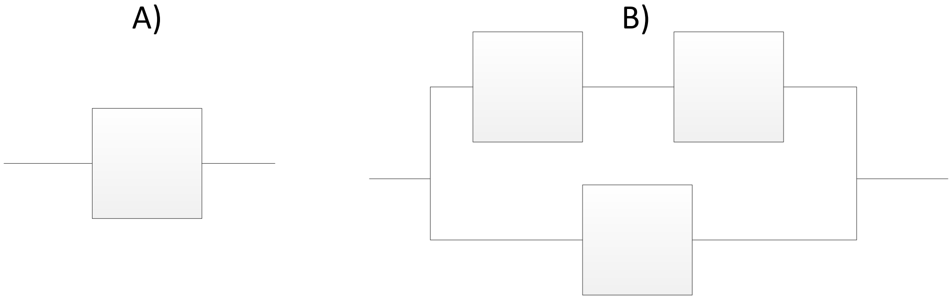
PECSN, 28/7/21

Exercise 1

ACME components owns two switch production plants. In plant 1, each unit is defective with probability $p_1 = 10^{-5}$, independently from the others. In plant 2, the mean weekly number of defective units is equal to 5. The production of each plant is $n = 4 \cdot 10^5$ units per week.

1) Compute mean and variance of the number of defective units produced by ACME in a week.
2) Draw a *qualitative* plot (with as many details are possible) of the PMF of the number of defective units in a week.
3) Compute the probability that the weekly number of defective units produced by ACME is equal to 5.
4) Compute the probability that the weekly number of defective units produced by ACME is less than 3.
5) Compute the probability that a randomly chosen unit is defective.

A) B)



Suppose now that ACME units can be connected in series or in parallel as above, and that the resulting system works if there exists a way that connects both extremities traversing only non-defective systems.

6) Explain which of the two systems has a higher chance to be functioning. Justify your findings.

Exercise 2

A network buffer has enough space for three packets. It employs a *gated* policy, meaning that it only accepts ingresses when the system is *empty*. Ingresses come in the form of *messages*, each one containing *one*, *two or three* packets, with probability q_1, q_2, q_3 respectively. The interarrival time of messages is an exponentially distributed variable with a mean equal to $\frac{1}{\lambda}$. The buffer processes *packets* (not messages), and the service time of a packet is an exponentially distributed variable with a mean equal to $\frac{1}{\mu}$.

1) Model the system and draw the CTMC
2) Compute the steady-state probabilities and the stability condition
3) Determine which number of packets in the system is the most likely at the steady state
4) Compute the mean number of packets in the system and in the queue. State the conditions under which the mean number of packets in the system is larger than one.
5) Compute the system utilization.

Exercise 1 - Solution

1) Given that n is large and p is small, we can approximate the failure probability of each plant using a Poisson variable, whose average is $\lambda_i = n_i \cdot p_i$. Hence, it is $\lambda_1 = 4, \quad \lambda_2 = 5$. Thus, there are on average 9 defective units in a weekly production of $2n = 8 \cdot 10^5$ pieces. As for the variance, it is all the more reasonable to approximate the whole production using a Poisson variable, whose average and variance is equal to 9.
2) The Poisson variable has a bell shape, with an infinite right tail. It peaks around its mean value, which is equal to 9.

3) The probability that 5 pieces are defective is equal to $p_5 = e^{-9} \cdot \frac{9^5}{5!} = 0.060727$
4) The probability that less than 3 pieces are defective is equal to $p_0 + p_1 + p_2 = 1.23 \cdot 10^{-4} + 1.11 \cdot 10^{-3} + 4.998 \cdot 10^{-3} = 6.232 \cdot 10^{-3}$
5) The probability is the following:

$$\begin{aligned} p_d &= P\{\text{defective}\} \\ &= P\{\text{defective}|\text{plant 1}\} \cdot P\{\text{plant 1}\} + P\{\text{defective}|\text{plant 2}\} \cdot P\{\text{plant 2}\} \\ &= 10^{-5} \cdot 0.5 + \frac{5}{4 \cdot 10^5} \cdot 0.5 \\ &= 1.125 \cdot 10^{-5} \end{aligned}$$

6) System a) works with probability $p_a = 1 - p_d$. System b) works with probability

$$\begin{aligned} p_b &= 1 - P\{\text{upper branch fails}\} \cdot P\{\text{lower branch fails}\} \\ &= 1 - (1 - (1 - p_d)^2) \cdot p_d \end{aligned}$$

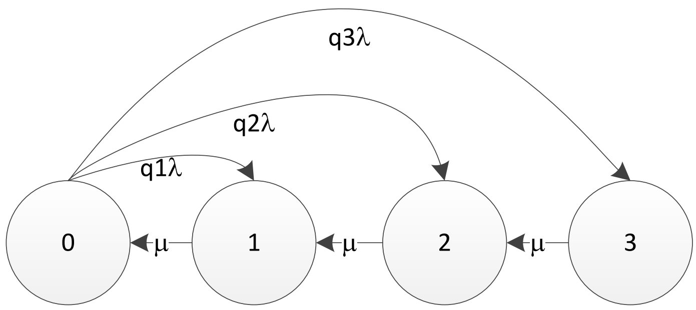
Thus, $p_b > p_a$ if and only if

$$\begin{aligned} &1 - (1 - (1 - p_d)^2) \cdot p_d > 1 - p_d \\ &p_d > (1 - (1 - p_d)^2) \cdot p_d \\ &1 > 1 - (1 - p_d)^2 \\ &p_d < 1 \end{aligned}$$

which is always true. System b) is always more reliable than system a), no matter what the failure probability of a single component is.

Exercise 2 – Solution

1) The CTMC is as follows. Note that $q_1 + q_2 + q_3 = 1$, obviously.



2) The steady state probabilities are computed by writing down the global equilibrium equations:

$$\begin{aligned} P_0 \cdot (q_1 + q_2 + q_3) \cdot \lambda &= P_1 \cdot \mu \\ P_1 \cdot \mu &= P_2 \cdot \mu + P_0 \cdot q_1 \cdot \lambda \\ P_2 \cdot \mu &= P_3 \cdot \mu + P_0 \cdot q_2 \cdot \lambda \\ P_3 \cdot \mu &= P_0 \cdot q_3 \cdot \mu \end{aligned}$$

One of the above equations is redundant. The system is always stable, since it has a finite queue. By solving the above system, we obtain:

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (q_1 + 2q_2 + 3q_3)} = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (1 + q_2 + 2q_3)}$$

$$P_1 = \frac{\lambda}{\mu} \cdot P_0 = \frac{\lambda}{\mu} \cdot (q_1 + q_2 + q_3) P_0$$

$$P_2 = \frac{\lambda}{\mu}(1 - q_1) \cdot P_0 = \frac{\lambda}{\mu}(q_2 + q_3) \cdot P_0$$

$$P_3 = \frac{\lambda}{\mu} q_3 \cdot P_0$$

3) From the above, it is clear that $P_1 \geq P_2 \geq P_3$, whatever the values q_i. Whether $P_0 > P_1$ or vice versa instead depends on whether $\lambda > \mu$ or vice versa. Thus, the answer is P_0 if $\lambda > \mu$, and P_1 otherwise.
4) It is:

$$E[N] = \sum_{n=1}^{3} n \cdot P_n = \frac{\lambda}{\mu} \cdot P_0 \cdot [1 \cdot (q_1 + q_2 + q_3) + 2(q_2 + q_3) + 3q_3]$$

$$= \frac{q_1 + 3q_2 + 6q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{1 + 2q_2 + 5q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$

$$E[N_q] = 1 \cdot P_2 + 2 \cdot P_3 = \frac{\lambda}{\mu} \cdot P_0 \cdot (q_2 + 3q_3) = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$

In order to have $E[N] > 1$ we need to have $q_1 + 3q_2 + 6q_3 > \frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3$, which translates to
$q_2 + 3q_3 > \frac{\mu}{\lambda}$.

The system utilization is $1 - P_0$, i.e. $U = \frac{q_1+2q_2+3q_3}{\frac{\mu}{\lambda}+q_1+2q_2+3q_3} = \frac{1+q_2+2q_3}{\frac{\mu}{\lambda}+1+q_2+2q_3}$.