PECSN 6/7/2021

Exercise 1

Consider the following function:

$$f(x)=\begin{cases}\frac{k}{c}\cdot\left(\frac{x}{c}\right)^{k-1} & 0\le x\le c\\ 0 & otherwise\end{cases}.$$

1) Express the conditions under which $f(x)$ is a PDF
2) Compute the CDF $F(x)$, draw it for $k=1$ and $k=2$, and discuss what happens when $k\to\infty$.
3) Compute the mean value and the median. Justify the result when $k=1$ and $k\to\infty$.
4) Compute the variance. Justify the result.
5) Consider RV $S_j=\sum_{i=1}^{j}X_i$, where the X_i s are IID distributed according to $f(x)$. Compute the coefficient of variation *C* of S_j (CoV: $C=\sigma/E[S_j]$). Justify the result.

Exercise 2

A car repair service company has *n* repair bays, and expects customers' cars to come in for repair with exponentially distributed interarrivals, at a rate λ. The repair of a car takes an exponentially distributed time with a mean of $1/\mu$. The company wants to man the smallest possible number of repair bays (so as to save money), but knows that its customers find it unacceptable to have to wait.

1) Model the above system as a birth-death process and draw the transition diagram (CTMC)
2) Compute the steady-state probabilities. Express the stability condition
3) Compute the probability P_{wait} that a car that breaks has to *wait* before entering a repair bay
4) Assume $\lambda=\mu$. Compute P_{wait} as a function of *n* and study its behavior with *n*.
5) Under the above hypothesis, state whether 6 manned repair bays are enough to have P_{wait} smaller than $5\cdot10^{-4}$.

It may be useful to observe that $\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}=\lim_{n\to\infty}\left[\sum_{j=0}^{n}\frac{x^j}{j!}\right]_{x=1}=\left[e^x\right]_{x=1}=e$, and that $\sum_{j=0}^{n}\frac{1}{j!}\approx e$ when $n\ge5$.

Exercise 1 - Solution

1) The conditions that must hold are that:

a) the function must be positive, hence $k > 0$

b) its integral must be equal to 1, i.e.: $\int_{-\infty}^{+\infty} f(x)dx = \frac{k}{c^k} \cdot \int_0^c x^{k-1}dx = \frac{k}{c^k} \cdot \left[\frac{x^k}{k} \right]_0^c = 1$.

Therefore, the only required condition is that $\boxed{k > 0}$.

2) The CDF can be easily obtained as: $\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{c} \right)^k & 0 \le x \le c \\ 1 & x > c \end{cases}}$.

As *k* grows large, the CDF tends to become more and more similar to a step function in $x = c$. Note that this can be interpreted as the distribution of the *maximum* of *k* IID RVs ~U(0,*c*), as seen during classes.

3) The mean value is $\boxed{E[X] = \int_0^c x \cdot f(x)dx = \int_0^c k \cdot \left(\frac{x}{c} \right)^k dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+1}}{k+1} \right]_0^c = \frac{k \cdot c}{k+1}}$.

The median is instead obtained by equating $F(x) = 0.5$, i.e., $\left(\frac{x}{c} \right)^k = \frac{1}{2}$, i.e. $\boxed{x_{0.5} = \frac{c}{\sqrt[k]{2}}}$. When $k = 1$ the distribution is uniform, hence symmetric, thus $E[X] = x_{0.5}$. As *k* grows large, it is again $E[X] = x_{0.5} = c$, since the distribution becomes a step function.

4) The mean square value is: $E\left[X^2\right] = \int_0^c x^2 \cdot f(x)dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+2}}{k+2} \right]_0^c = \frac{k \cdot c^2}{k+2}$.

Hence the variance is: $\boxed{\sigma^2 = E\left[X^2\right] - E[X]^2 = \frac{k \cdot c^2}{k+2} - \frac{k^2 \cdot c^2}{(k+1)^2} = \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}}$.

The variance depends on the square of *c*, since the latter is the range of the variable. Moreover, it decreases with *k*. This makes perfect sense, since, as *k* grows large, the variability of the distribution decreases as well, as already observed. When $k = 1$, we obtain again the well-known variance of a uniform variable.

5) The mean and variance are $j \cdot \frac{k \cdot c}{k+1}$ and $j \cdot \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}$ respectively. Thus, it is:

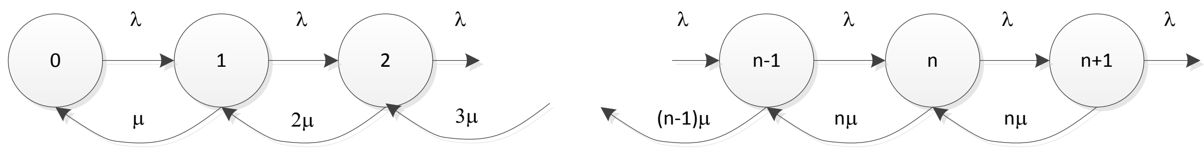
$$C=\frac{\sqrt{j\cdot\frac{k\cdot c^2}{(k+2)\cdot(k+1)^2}}}{j\cdot\frac{k\cdot c}{k+1}}=\frac{\sqrt{j\cdot\frac{k}{(k+2)}}}{j\cdot k}=\frac{1}{\sqrt{j\cdot k\cdot(k+2)}}$$

The result is in accord with the Central Limit Theorem, which is applicable since variables are IID with finite mean and variance. The CoV decreases with the number of variables, and it decreases *faster* if *k* is large. In fact, as *k* grows large, the mean converges, but the variance goes to zero. When *k* is large, the following approximation holds:

$$\boxed{C=\frac{1}{\sqrt{j\cdot k\cdot(k+2)}}\approx\frac{1}{k}\cdot\frac{1}{\sqrt{j}}}.$$

Exercise 2 - Solution

1) The system is an *M*/*M*/*n* one, hence the diagram is the following:



2) We know from the theory that the system is stable if $\rho=\lambda/(n\cdot\mu)<1$. This should also emerge from the computation of the steady-state probabilities. The global equilibrium equations are the following:

$$\begin{aligned}
&P_0\cdot\lambda=P_1\cdot\mu\\
&P_1\cdot\lambda=P_2\cdot 2\mu\\
&\ldots\\
&P_{n-1}\cdot\lambda=P_n\cdot n\cdot\mu\\
&P_n\cdot\lambda=P_{n+1}\cdot n\cdot\mu\\
&\ldots\\
&P_{n+j}\cdot\lambda=P_{n+j+1}\cdot n\cdot\mu \quad j\geq 0
\end{aligned}$$

From which we get:

$$P_j=\begin{cases}\left(\frac{\lambda}{\mu}\right)^j\cdot\frac{1}{j!}\cdot P_0 & j<n\\ \rho^j\cdot\frac{n^n}{n!}\cdot P_0 & j\geq n\end{cases}$$

Hence, the normalization condition is:

$$P_0\cdot\left\{\sum_{j=0}^{n-1}\left[\left(\frac{\lambda}{\mu}\right)^j\cdot\frac{1}{j!}\right]+\frac{n^n}{n!}\cdot\sum_{j=n}^{\infty}\rho^n\right\}=1$$

The infinite sum converges if and only if $\rho < 1$, as expected. This said,

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \left[\sum_{j=0}^{\infty} \rho^n - \sum_{j=0}^{n-1} \rho^n \right] \right\} = 1$$

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\} = 1 \quad .$$

$$P_0 = \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\}^{-1}$$

If *n* is large, the following approximation is reasonable: $P_0 = \dfrac{1}{e^{\lambda/\mu} + \dfrac{1}{n!} \cdot \dfrac{(n \cdot \rho)^n}{1-\rho}}$

3) Since the system enjoys the PASTA property, the probability that a car that breaks has to wait before entering service is the probability that $j \geq n$ customers are in the system, i.e. $\sum_{j=n}^{\infty} r_j = \sum_{j=n}^{\infty} P_j$. This can be written as:

$$P_{wait} = \sum_{j=n}^{\infty} P_j = \frac{n^n}{n!} \cdot P_0 \cdot \sum_{j=n}^{\infty} \rho^j = \frac{(n \cdot \rho)^n}{n!(1-\rho)} \cdot P_0 = \frac{\dfrac{(n \cdot \rho)^n}{n!(1-\rho)}}{\sum_{j=0}^{n-1} \left[\left(\dfrac{\lambda}{\mu} \right)^j \cdot \dfrac{1}{j!} \right] + \dfrac{(n \cdot \rho)^n}{n!(1-\rho)}} .$$

Again, if *n* is large, the following approximation is reasonable:

$$P_{wait} \approx \frac{1}{\dfrac{n!(1-\rho) \cdot e^{\lambda/\mu}}{(n \cdot \rho)^n} + 1}$$

4) Note that $\lambda = \mu$ implies $n > 1$, otherwise the system is unstable. When $\lambda = \mu$, we get:

$$P_{wait} = \frac{\dfrac{\left(n \cdot \dfrac{1}{n} \right)^n}{n!\left(1 - \dfrac{1}{n} \right)}}{\sum_{j=0}^{n-1} \left[\dfrac{1}{j!} \right] + \dfrac{\left(n \cdot \dfrac{1}{n} \right)^n}{n!\left(1 - \dfrac{1}{n} \right)}} = \frac{1}{(n-1)!(n-1) \cdot \sum_{j=0}^{n-1} \left[\dfrac{1}{j!} \right] + 1} .$$

The above statement is confirmed by the fact that $(n-1)$ appears in the denominator.

P_{wait} is obviously decreasing with *n* (since the denominator increases with *n*). Moreover, $P_{wait}(n) \geq \dfrac{1}{(n-1)!(n-1) \cdot e + 1}$, with $P_{wait}(n) \approx \dfrac{1}{(n-1)!(n-1) \cdot e + 1}$ when $n \geq 5$. The first few values are reported in the table:

n	Pwait expr.	Numerical value
2	$1/3$	0.333333
3	$1/11$	0.090909
4	$1/49$	0.020408
5	$1/261$	0.003831

5) When $n = 6$, it is $P_{wait}(n) \approx \frac{1}{600 \cdot e + 1}$. Since $e < 3$, it is $P_{wait}(n) > \frac{1}{1800} > \frac{1}{2000} = 5 \cdot 10^{-4}$, so the answer is no.