PECSN, 23/2/21

Exercise 1

Two bank clerks are assigned *standard* and *urgent* customers respectively. Let X denote the number of customers being attended to by the first clerk, and Y denote the number of customers of the second one at the same time. Let the JPMF of X and Y be the following:

X / Y	0	1	2	3
0	0.08	0.07	0.04	0.00
1	0.06	0.15	0.05	0.04
2	0.05	0.04	0.10	0.06
3	0.00	0.03	0.04	0.07
4	0.00	0.01	0.05	0.06

1) What is the probability that there is exactly one customer in each line?

2) What is the probability that the number of customers in the two lines are identical?

3) Let A denote the event that there are at least two more customers in one line than in the other line. Calculate the probability of A.

4) Determine the marginal PMF of X and then calculate the expected number of standard customers in line.

5) Determine the marginal PMF of Y.

6) Are X and Y independent random variables? Explain your answer.

7) Determine the PMF of the *overall* number of customers in line at the bank.

Exercise 2

Consider a single-server queueing system where the arrival and departure rates are ($k \geq 0$):

$$\lambda_j = \begin{cases} \frac{\lambda}{(j+1)} & 0 \leq j < k \\ \lambda & j \geq k \end{cases}, \qquad \mu_j = \begin{cases} \mu & 1 \leq j \leq k \\ j \cdot \mu & j > k \end{cases}.$$

1) Draw the CTMC.
2) Compute the stability condition and the steady-state probabilities. State explicitly whether and how both depend on k.
3) Express the condition by which s.s. probabilities are a decreasing sequence.
4) Compute the mean number of jobs in the system and in the queue. State explicitly how both depend on k.
5) Compute the mean arrival rate as a function of k. Discuss what happens when $k = 0, k \to +\infty$ and justify your results.
6) Compute the mean response time when $k = 0, k \to +\infty$.

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Exercise 1 - Solution

1) What is the probability that there is exactly one customer in each line?

$$P(X=1, Y=1) = p(1,1) = 0.15$$

2) What is $P(X = Y)$, that is, the probability that the number of customers in the two lines are identical?

$$P(X=Y) = p(0,0) + p(1,1) + p(2,2) + p(3,3) = 0.08 + 0.15 + 0.1 + 0.07 = 0.4$$

3) Let A denote the event that there are at least two more customers in one line than in the other line. Calculate the probability of A.

$$A = \{(x,y): x \geq y+2\} \cup \{(x,y): y \geq x+2\}$$
$$= \{(2,0),(3,0),(4,0),(3,1),(4,1),(4,2),(0,2)(0,3)(1,3)\}$$

$$P(A) = p(2,0) + p(3,0) + p(4,0) + p(3,1) + p(4,1) + p(4,2) + p(0,2) + p(0,3) + p(1,3)$$
$$= 0.22$$

4) Determine the marginal PMF of X and then calculate the expected number of customers in the standard queue.

$$p_X(n) = \sum_{i=-\infty}^{+\infty} p(n,i) = p(n,0) + p(n,1) + p(n,2) + p(n,3)$$

x	0	1	2	3	4
$p_X(x)$	0.19	0.30	0.25	0.14	0.12

Hence, $E(X) = \sum_{x=1}^{4} x \cdot p_X(x) = 1 \cdot 0.19 + 2 \cdot 0.25 + 3 \cdot 0.14 + 4 \cdot 0.12 = 1.7$

5) Determine the marginal PMF of Y .

$$p_Y(n) = \sum_{i=-\infty}^{+\infty} p(i,n) = p(0,n) + p(1,n) + p(2,n) + p(3,n) + p(4,n)$$

y	0	1	2	3
$p_Y(y)$	0.19	0.30	0.28	0.23

6) Are X and Y independent random variables? Explain.

They are not. By counterexample: $P(X=4) = 0.12$, $P(Y=0) = 0.19$, $P(X=4, Y=0) = 0$.

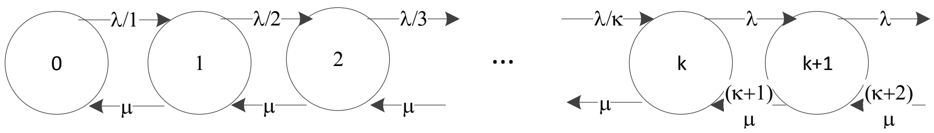
7) Determine the PMF of the *overall* number of customers in line at the bank.

The PMF is obtained by summing up all the values (x,y) having the same sum $s = x + y$, i.e.

s	$p(s)$
0	0.08
1	0.13
2	0.24
3	0.09
4	0.17
5	0.11
6	0.12
7	0.06

Exercise 2 - Solution

1) The CTMC is the following:



2) Call $u = \frac{\lambda}{\mu}$. The local equilibrium equations are always $P_{j+1} = \frac{u}{j+1} \cdot P_j$, regardless of whether $j < k$ or $j \geq k$, hence $P_j = \frac{u^j}{j!} \cdot P_0, j \geq 0$. Therefore, the system is always stable, and the s.s. probabilities are the same as those of a discouraged arrival system or an M/M/∞ one, i.e. $P_j = e^{-u} \frac{u^j}{j!}, \forall j, \geq 0$. The s.s. probabilities and the stability condition do not depend on k.

3) The s.s. probabilities are a Poisson distribution with a parameter u. We know from the theory that the Poisson PDF is monotonically decreasing if $u < 1$. The same result can be obtained by imposing that $\forall j, P_j > P_{j+1}$, which yields $\forall j, j+1 > u$, hence $u < 1$.

4) From the theory on Poisson distribution, we readily obtain $E[N] = u$. Moreover, in every single-server system it is: $E[N_q] = E[N] - (1 - P_0)$, hence $E[N_q] = u - (1 - e^{-u})$. Neither of the above depend on k.

5) The system is non-PASTA, since the arrival rate depends on the state. It is, in fact:

$\bar{\lambda} = \lambda \cdot \left[\sum_{j=0}^{k-1} \frac{1}{j+1} e^{-u} \frac{u^j}{j!} + \sum_{j=k}^{+\infty} e^{-u} \frac{u^j}{j!}\right] = \lambda \cdot e^{-u} \cdot \left[\frac{1}{u} \cdot \left(\sum_{j=0}^{k} \frac{u^j}{j!} - 1\right) + e^u - \sum_{j=0}^{k} \frac{u^j}{j!} + \frac{u^k}{k!}\right] = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1\right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!}\right]$,

where we use $S_k = \sum_{j=0}^{k} \frac{u^j}{j!}$ for the sake of conciseness. Note that $S_0 = 1$, and $\lim_{k \to +\infty} S_k = e^u$.

The mean arrival rate does depend on k. We deal with the two cases $k = 0, k \to +\infty$ separately:

a) $k = 0$. In this case, the arrival rate is constant and equal to λ. In fact, from the above formula we get:

$$\bar{\lambda} = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1\right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!}\right] = \lambda \cdot e^{-u} \cdot \left[\frac{1}{u} - 1 + e^u - \frac{1}{u} + 1\right] = \lambda$$

b) $k \to +\infty$. In this case, the mean arrival rate is the one of a discouraged arrival system. In fact we get:

$$\bar{\lambda} = \lim_{k \to +\infty} \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1 \right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!} \right] = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1 \right) e^u + e^u - \frac{1}{u} \right]$$
$$= \mu \cdot [1 - e^{-u}]$$

6) We deal with the two cases $k = 0, k \to +\infty$ separately:

a) $k = 0$. In this case we get $E[R] = \frac{E[N]}{\bar{\lambda}} = \frac{u}{\lambda} = \frac{1}{\mu}$. This makes sense, since when $k = 0$ the system is akin to a load-dependent server with infinite capacity.

b) $k \to +\infty$. In this case we get $E[R] = \frac{E[N]}{\bar{\lambda}} = \frac{u}{\mu \cdot [1 - e^{-u}]} = \frac{\lambda}{\mu^2 \cdot [1 - e^{-u}]}$. This is in fact the mean response time of a discouraged-arrival system.